Revealed smooth nontransitive preferences

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Revealed smooth nontransitive preferences*

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Abstract

In the present paper, we are concerned with the behavioural consequences of consumers having nontransitive preference relations. Data sets consist of finitely many observations of price vectors and consumption bundles. A preference relation rationalizes a data set provided that for every observed consumption bundle, all strictly preferred bundles are more expensive than the observed bundle. Our main result is that data sets can be rationalized by a smooth nontransitive preference relation if and only if prices can normalized such that the law of demand is satisfied. Market data sets consist of finitely many observations of price vectors, lists of individual incomes and aggregate demands. We apply our main result to characterize market data sets consistent with equilibrium behaviour of pure-exchange economies with smooth nontransitive consumers.

Keywords: law of demand, revealed preferences, GARP, SARP, SSARP, WARP.

JEL-classification: D1, D5.

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1 Introduction

In the present paper, we are concerned with the behavioural consequences of consumers having nontransitive (complete, but not necessarily transitive) preference relations. On the one hand, in Quah (2006) it is shown that a demand function satisfies the Weak Axiom of Revealed Preferences (WARP) if and only if it is identical to the behaviour of a nontransitive consumer. On the other hand, for data sets consisting of finitely many observations of price vectors and consumption bundles, there are data sets unextendable to demand functions satisfying WARP even though the data sets satisfy WARP. Here we characterize data sets consistent with the behaviour of smooth nontransitive consumers. Below we elaborate.

In Samuelson (1938, 1948) WARP is introduced. In case of two goods, a demand function is shown to satisfy WARP if and only if it is consistent with utility maximizing behaviour. The Strong Axiom of Revealed Preferences (SARP) is introduced in Houthakker (1950), where it is shown that in case of a finite number of goods, a demand function satisfies SARP if and only if it is consistent with utility maximizing behaviour. Consequently, in case of at least three goods, WARP is necessary, but not sufficient, for a demand function to be consistent with utility maximizing behaviour.

In Afriat (1967) data sets rather than demand functions are considered. It is shown that a data set is consistent with utility maximizing behaviour if and only if it satisfies cyclical consistency. An algorithm for checking whether data sets satisfies cyclical consistency is provided in Diewert (1973). In Varian (1982) the Generalized Axiom of Revealed Preferences (GARP) is introduced. It is shown that a data set is consistent with utility maximizing behaviour if and only if it satisfies GARP. Moreover it is shown that cyclical consistency and GARP are equivalent. Intuitively, the difference between GARP and SARP is that GARP allows convex preference relations, while SARP allows only strictly convex preference relations.

In Chiappori & Rochet (1987) the Strong version of the Strong Axiom of Revealed Preferences (SSARP) is introduced. A data set is shown to be consistent with utility maximizing behaviour for a smooth utility function (satisfying the standard assumptions used to ensure differentiability of demand functions) if and only if it satisfies SSARP. However, the utility function is restricted to a compact set of consumption bundles. Intuitively, the difference between SARP and SSARP is that SARP allows indifference sets with kinks, while SSARP allows only indifference sets without kinks.

Afriat's theorem is extended to general budget set in Forges & Iehlé (2013) and Forges & Minelli (2009), where general budget sets are compact and monotonic sets. Afriat's theorem is interpreted as a second welfare theorem for the housing allocation problem of Shapley and Scarf in Ekeland & Galichon (2013).

In Sonnenschein (1971) and Shafer (1974) nontransitive preference relations are considered. It is shown that strict convexity and continuity imply that for every pair of a price vector and an income there is a unique maximal element. Hence, there is a demand function and it is continuous. Demand functions satisfying WARP and SARP, respectively, are compared in Kihlstrom, Mas-Colell & Sonnenschein (1976). For differentiable demand functions it is shown that: (1) a demand function satisfies WARP if and only if all Slutsky matrices are negative semidefinite, but not necessarily symmetric; and, (2) a demand function satisfies SARP if and only if all Slutsky matrices are negative semidefinite and symmnetric. In Al-Najjar (1993) smooth nontransitive preference relations are considered. It is shown that for these preference relations, consumers have smooth demand functions.

A class of convex nontransitive preference relations is considered in Quah (2006). A demand function is shown to satisfy WARP if and only if it is the demand function associated with some preference relation in the considered class of preference relations. Consequently, WARP is useful for characterizing demand functions rationalizable by convex nontransitive preference relation. A preference relation rationalizes a data set provided that for every observed consumption bundle, all strictly preferred bundles are more expensive than the observed bundle. In John (2001) rationalizability of data sets by nontransitive preference relations is considered. It is shown that data sets can be rationalized by continuous, nontransitive, monotonic and convex preference relations if and only if prices can be normalized such that the law of demand for correspondences is satisfied. For these preference relations, consumers have demand correspondences rather than demand functions.

In Brown & Matzkin (1996) market data sets consisting of finitely many observations of price vectors, lists of individual incomes and aggregate demands are considered. Therefore consistency of market data sets reduces to whether aggregate demand can be split into individual consumption bundles such that individual data sets consisting of pairs of price vectors and consumption bundles are consistent with utility maximizing behaviour. A market data set is shown to be consistent with equilibrium behaviour for some pure-exchange economy with utility maximizing consumers if and only if there are consumption bundles such that markets clear and for every consumer, budget constraints and GARP are satisfied. The result implies that the general equilibrium model has testable implications. An analogous result for pure-exchange economies with utility maximizing consumers with smooth utility functions (satisfying the standard assumptions used to ensure differentiability of demand functions) is presented in Balasko & Tvede (2010b).

In the present paper we consider rationalizability of data sets by smooth nontransitive preference relations. Our main result is that data sets can be rationalized by smooth, nontransitive, strongly monotonic and strictly convex preference relations if and only if prices can be normalized such that the law of demand is satisfied. Our main result can be seen as a smooth version of the result in John (2001) in the same way as the result in Chiappori & Rochet can be seen as a smooth version of the result in Afriat (1967). We apply our main result to characterize market data sets consistent with equilibrium behaviour of some pure-exchange economy with smooth nontransitive consumers.

Moreover we consider two examples of data sets. Firstly, a data set satisfying WARP but not rationalizable by any convex nontransitive preference relation. Consequently, WARP is useless for characterizing data sets rationalizable by convex nontransitive preference relations. To the best of our knowledge, this is the first example of a data set satisfying WARP, yet not rationalizable by any convex nontransitive preference relation. Secondly, a data set satisfying WARP, for which prices cannot be normalized such that the law of demand for correspondences is satisfied. However, we show that the data set can be rationalized by a weakly continuous, nontransitive, monotonic and convex preference relation, for which the associated demand correspondence satisfies WARP for correspondences. The example shows that there are weakly continuous preference relations for which the associated demand correspondences satisfy WARP for correspondences, but prices cannot be normalized such that the law of demand for correspondences is satisfied.

The paper is organized as follows: in Section 2 the set up is presented and WARP and GARP are stated; in Section 3 the main result on rationalizability of data sets by smooth nontransitive preference relations is stated; Section 4 contains the proof of the main result; in Section 5 the condition in the main result is shown to be much too strong to characterize all data sets rationalizable by preference relations for which the associated demand correspondences satisfy WARP for correspondences; in Section 6 the main result is applied to characterize market data sets consistent with equilibrium behaviour for some pure-exchange economy with smooth nontransitive consumers; and, finally Section 7 contains some concluding remarks.

2 The set up

In the present section, we introduce data sets, preference relations and two axioms of revealed preferences.

Data sets, preference relations and rationalizability

A data set $\mathscr{D}$ consists of finitely many observations of pairs of price vectors and consumption bundles

$$\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$$

where $p_i \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $x_i \in \mathbb{R}^\ell_+$ for all i.

A preference relation is a set $\mathscr{P} \subseteq \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$. Let $x \mathscr{P} y$ denote $(x,y) \in \mathscr{P}$. The strict preference relation $\mathscr{P}^\circ$ associated with $\mathscr{P}$ is defined by

$$x \mathscr{P}^\circ y \iff [x \mathscr{P} y \text{ and } \neg(y \mathscr{P} x)].$$

The following properties of preference relations are considered in the sequel:

Completeness: $x \mathscr{P} y$ or $y \mathscr{P} x$ for all x and y.

Transitivity: $x \mathscr{P} y$ and $y \mathscr{P} z$ imply $x \mathscr{P} z$ for all x, y and z.

Local non-satiatedness: For all x and $\varepsilon > 0$ there is y with $\|y-x\| < \varepsilon$ and $y \mathscr{P}^\circ x$.

Monotonicity: $x \in \{y\} + \mathbb{R}^\ell_{++}$ implies $x \mathscr{P}^\circ y$ for all x and y.

Strong monotonicity: $x \in \{y\} + \mathbb{R}^\ell_+ \setminus \{0\}$ implies $x \mathscr{P}^\circ y$ for all x and y.

Convexity: $x \mathscr{P} y$ and $z \mathscr{P} y$ imply $((1-t)x + tz) \mathscr{P} y$ for all x, y and z and $t \in [0,1]$.

Strict convexity: $x \mathscr{P} y$ and $z \mathscr{P} y$ imply $((1-t)x + tz) \mathscr{P}^\circ y$ for all x, y and z with $x \neq z$ and $t \in]0,1[$.

Continuity: The sets $\{x \mid x \mathscr{P} y\}$ and $\{z \mid y \mathscr{P} z\}$ are closed for all y.

All properties are standard.

Using the terminology in Shafer (1974), complete, but not necessarily transitive, preference relations are denoted *nontransitive preference relations*.

Definition 1 *A preference relation $\mathscr{P}$* **rationalizes** *a data set $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$ provided that for all x_i and y, $y \mathscr{P}^\circ x_i$ implies $p_i \cdot (x_i - y) < 0$.*

All data sets can be rationalized by the preference relations $\mathscr{P}$ and $\mathscr{Q}$, where $x \mathscr{P} y$ and $y \mathscr{P} x$ for all x and y and $x \mathscr{Q}^\circ y$ if and only if $x \in \{y\} + \mathbb{R}^\ell_{++}$.

The Strong and Weak Axioms of Revealed Preferences

A data set $\mathscr{D}$ satisfies SARP provided that for every subset of observations $\{(p_{i_1},x_{i_1}),\ldots,(p_{i_k},x_{i_k})\}$,

$$p_{i_1} \cdot (x_{i_1} - x_{i_2}) \geq 0,\, p_{i_2} \cdot (x_{i_2} - x_{i_3}) \geq 0, \ldots,\, p_{i_{k-1}} \cdot (x_{i_{k-1}} - x_{i_k}) \geq 0$$

imply $x_{i_1} = x_{i_k}$ or $p_{i_k} \cdot (x_{i_k} - x_{i_1}) < 0$. In Afriat (1967) and Varian (1982) it is shown that a data set satisfies SARP if and only if it can be rationalized by a preference relation that can be represented by a monotonic, strictly quasi-concave and continuous utility function.

A data set $\mathscr{D}$ satisfies WARP provided that for every pair of observations i and j, $p_i \cdot (x_i - x_j) \geq 0$ implies $x_i = x_j$ or $p_j \cdot (x_j - x_i) < 0$. For a preference relation $\mathscr{P}$ the associated demand function $\phi_{\mathscr{P}}$ is defined by $x = \phi_{\mathscr{P}}(p,w)$ provided $\{y \mid p \cdot y \leq w\} \cap \{z \mid z \mathscr{P} x\} = \{x\}$. A demand function $\phi_{\mathscr{P}}$ satisfies WARP provided that all pairs of observations (p_1,x_1) and

(p_2, x_2) with $x_i = \phi_{\mathscr{P}}(p_i, p_i \cdot x_i)$ for both i satisfy WARP. A preference relation $\mathscr{P}$ satisfies WARP provided the associated demand function $\phi_{\mathscr{P}}$ satisfies WARP.

As explained in the introduction, the connection between nontransitive preference relations and WARP is studied in Kihlstrom, Mas-Colell & Sonnenschein (1976) and Quah (2006). In Quah (2006) preference relations satisfying completeness, strong monotonicity and weak forms of strict convexity and continuity are considered: it is shown that a demand function satisfies WARP if and only if it is the demand function associated with a preference relation satisfying these conditions. However, going from demand functions to data sets is not straightforward. Indeed there are data sets satisfying WARP not rationalizable by any preference relation satisfying WARP as the following example shows.

Example 1: Consider a data set consisting of three observations $\{(p_1, x_1), (p_2, x_2)(p_3, x_3)\}$ with

$$\begin{aligned} p_1 &= (4,1,5) & x_1 &= (4,1,1) \\ p_2 &= (5,4,1) & x_2 &= (1,4,1) \\ p_3 &= (1,5,4) & x_3 &= (1,1,4). \end{aligned}$$

Then

$$\begin{aligned} p_1 \cdot (x_1 - x_2) &= p_2 \cdot (x_2 - x_3) &= p_3 \cdot (x_3 - x_1) &= 9 \\ p_2 \cdot (x_2 - x_1) &= p_3 \cdot (x_3 - x_2) &= p_1 \cdot (x_1 - x_3) &= -3. \end{aligned}$$

Therefore the data set satisfies WARP. However there is no $p_4 \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$ for $x_4 = (1/3)(x_1 + x_2 + x_3) = (2,2,2)$ such that the data set $\{(p_1, x_1), (p_2, x_2)(p_3, x_3), (p_4, x_4)\}$ satisfies WARP. Indeed $p_i \cdot (x_i - x_4) = 2$ for all $i \in \{1,2,3\}$, but there is no p_4 such that $p_4 \cdot (x_4 - x_i) < 0$ for all $i \in \{1,2,3\}$. Therefore the data set cannot be rationalized by any convex preference relation satisfying WARP. *End of example*

Convexity of the consumption set is important in Example 1. Indeed a limited version of WARP is used to characterize data sets consistent with warm glow theory in a set up with finitely many alternatives in Cherepanov, Feddersen & Sandroni (2013).

Representation of nontransitive preference relations

In Shafer (1974) it is shown that completeness, strict convexity and continuity of preference relations ensure that demands are continuous functions. Therefore, for these preference relations: on the one hand, since preference relations are not necessarily transitive, there can be cycles of the form $x_1 \mathscr{P}^{\circ} x_2$, $x_2 \mathscr{P}^{\circ} x_3$ and $x_3 \mathscr{P}^{\circ} x_1$; and, on the other hand there is a unique demand for every price vector and income. Hence, if $p \cdot x_1, p \cdot x_2, p \cdot x_3 \leq w$, then there is x such that $p \cdot x \leq w$ and $x \mathscr{P}^{\circ} x_i$ for all i.

It turns out to be convenient to represent nontransitive preference relations by comparison functions.

Definition 2 *A function* $\Gamma : \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$ **represents** *a complete preference relation* $\mathscr{P}$ *provided that for all x and y,* $\Gamma(x,y) \geq 0$ *if and only if* $x \mathscr{P} y$ *and* $\Gamma(y,x) = -\Gamma(x,y)$.

According to Theorem 1 in Shafer (1974), every complete, strictly convex and continuous preference relation $\mathscr{P}$ can be represented by a continuous function $\Gamma : \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$.

Characterization of continuous nontransitive preference relations

In John (2001) the following characterization of the behavioural consequences of consumers having nontransitive preference relations is presented.

Theorem 1 *For a data set* $\mathscr{D}$ *the following conditions are equivalent:*

(i) $\mathscr{D}$ *can be rationalized by a complete and locally non-satiated preference relation that can be represented by a function that is concave in the first argument.*

(ii) For any set of real numbers $c_{ij} \geq 0$ *with* $c_{ij} = c_{ji}$ *the inequalities*

$$\sum_j c_{ij} p_i \cdot (x_j - x_i) \ \leq\ 0$$

for all i imply the equalities

$$\sum_j c_{ij} p_i \cdot (x_j - x_i) \ =\ 0$$

for all i.

(iii) There are real numbers $\lambda_1, \ldots, \lambda_n > 0$ *such that*

$$(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) \ \leq\ 0$$

for all i and j.

(iv) There are real numbers a_{ij}*,* $i, j \in \{1, \ldots, n\}$*, and* b_i*,* $i \in \{1, \ldots, n\}$*, with* $a_{ji} = -a_{ij}$ *for all i and j and* $b_i > 0$ *for all i such that*

$$a_{ij} \ \leq\ b_j p_j \cdot (x_i - x_j)$$

for all i and j.

(v) $\mathscr{D}$ *can be rationalized by a complete and monotonic preference relation that can be represented by a continuous function that is concave in the first argument.*

In Sonnenschein (1971) it is shown that if a preference relation is complete, monotonic, convex and continuous, then demand is an upper hemicontinuous correspondence.

Smooth nontransitive preference relations

In order to obtain smooth demand functions rather than upper hemicontinuous demand correspondences, additional assumptions on consumption sets and preference relations are considered. Consumption sets are assumed to be the interior of the positive orthant $\mathbb{R}^\ell_{++}$ rather than the positive orthant $\mathbb{R}^\ell_+$. Preference relations $\mathscr{P}$ are assumed to be represented by functions $\Gamma : \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \to \mathbb{R}$ with the following properties:

Smooth nontransitivity: Γ is smooth.

Smooth strong monotonicity: $D_1\Gamma(x,y) \in \mathbb{R}^\ell_{++}$ for all x and y.

Smooth strict concavity: $v^T D^2_{11}\Gamma(x,y)v < 0$ for all x and y and $v \in \mathbb{R}^\ell \backslash \{0\}$.

Closedness: $\{x \mid \Gamma(x,y) \geq 0\}$ is closed in $\mathbb{R}^\ell$ for all y.

Preference relations representable by functions with the four properties are denoted *smooth nontransitive preference relations*. In Balasko & Tvede (2010a) smooth nontransitivity, smooth strong monotonicity, smooth strict concavity and closedness of a preference relation $\mathscr{P}$ are shown to ensure that demand $\phi_{\mathscr{P}}$ is a smooth function.

In Sections 3, 4 and 6 consumption sets are assumed to be the interior of the positive orthant and smooth nontransitive preference relations are considered.

3 Rationalizability with the weighted law of demand

In the present section, we introduce the weighted law of demand and show that a data set can be rationalized by a smooth nontransitive preference relation if and only if the data set satisfies the weighted law of demand.

The weighted law of demand

A data set $\mathscr{D} = \{(p_1,x_1),\dots,(p_n,x_n)\}$ satisfies the Law of Demand provided that

$$(p_i - p_j)\cdot(x_i - x_j) < 0$$

for all i and j with $x_i \neq x_j$. It is well-known that a demand function ϕ satisfies WARP if and only if the law of demand holds for all pairs of observations (p_1,x_1) and (p_2,x_2) with $x_i = \phi(p_i, p_i\cdot x_i)$ for both i, $x_1 \neq x_2$ and $p_2\cdot x_2 = p_2\cdot x_1$.

A data set satisfies the *Weighted Law of Demand* (WLD) provided there is a normalization of prices such that the law of demand is satisfied.

Definition 3 *A data set $\mathscr{D} = \{(p_1,x_1),\dots,(p_n,x_n)\}$ satisfies WLD provided that*

(A.1) For every i, $p_i, x_i \in \mathbb{R}^\ell_{++}$.

(A.2) For every i and j, $x_i = x_j$ implies p_i and p_j are collinear.

(A.3) There are $\lambda_1, \ldots, \lambda_n > 0$ such that for all i and j, $x_i \neq x_j$ implies

$$(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) \ < \ 0.$$

(A.1) is necessary for smooth strong monotonicity and closedness of a preference relation rationalizing a data set. (A.2) is necessesary for smoothness of a preference relation rationalizing a data set. Indeed suppose that a data set $\mathscr{D}$ can be rationalized by a smooth nontransitive preference relation $\mathscr{P}$ represented by Γ, then for all i there is $\gamma_i > 0$ such that $D_1\Gamma(x_i, x_i) - \gamma_i p_i = 0$ as explained in Balasko & Tvede (2010a). (A.3) implies WARP. Indeed $(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) < 0$ and $p_i \cdot (x_i - x_j) \geq 0$ imply $p_j \cdot (x_j - x_i) < 0$. Moreover SARP implies (A.3). Indeed, according to Afriat (1967) and Varian (1982) a data set satisfies SARP if and only if there are $(U_1, \lambda_1), \ldots, (U_n, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n > 0$ such that for all i and j with $x_i \neq x_j$

$$U_j - U_i + \lambda_j p_j \cdot (x_i - x_j) \ > \ 0.$$

Adding the two inequalities for i and j gives (A.3). If p_i and p_j are collinear, then (A.3) implies that either $x_i = x_j$ or $p_i \cdot (x_i - x_j), p_j \cdot (x_i - x_j) \neq 0$ so either $x_i = x_j$ or $x_i - x_j$ and p_i or p_j are not orthogonal.

Characterization of rationalizable data sets

The following theorem justifies the introduction of WLD.

Theorem 2 *A data set $\mathscr{D}$ can be rationalized by a smooth nontransitive preference relation $\mathscr{P}$ if and only if the data set satisfies WLD.*

Remark: Theorem 2 is analogous to the theorem in Chiappori & Rochet (1987) and Theorem 2 in John (2001). Compared with Chiappori & Rochet (1987): we consider nontransitive preference relations defined on the whole consumption set rather than complete and transitive preference relations defined on a compact subset of the consumption set. Consequently demand is a smooth function for all prices and incomes rather than a smooth function for a compact set of prices and incomes. Compared with John (2001): we consider smooth nontransitive preference relations rather than preference relations represented by monotone, concave-convex (concave in the first coordinate and convex in the second coordinate) and continuous functions. Consequently demand is a smooth function rather than an upper hemicontinuous correspondence.

4 Proof of Theorem 2

The 'Rationalizability implies WLD' and the 'WLD implies Rationalizability' parts are both established through a series of lemmas.

For a data set $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$ it is assumed without loss of generality that $i \neq j$ implies $x_i \neq x_j$.

'Rationalizability implies WLD'

The proof consists of two lemmas.

Lemma 1 *Suppose that a data set $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$ can be rationalized by a locally non-satiated preference relation $\mathscr{P}$ that can be represented by a differentiable function Γ that is strictly concave in the first argument. Then for all $c_{11},c_{12},\ldots,c_{\ell\ell} \geq 0$ with $c_{ii} = 0$ for all i and $c_{ji} = c_{ji}$ for all i and j*

$$\sum_j c_{ij} p_i \cdot (x_j - x_i) \ \leq \ 0$$

for all i imply that $c_{ij} = 0$ for all i and j.

Proof: Suppose there are $c_{11},c_{12},\ldots,c_{\ell\ell} \geq 0$ with $c_{ii} = 0$ for all i and $c_{ji} = c_{ji}$ for all i and j, such that

$$\sum_j c_{ij} p_i \cdot (x_j - x_i) \ \leq \ 0$$

for all i and $\sum_{i,j} c_{ij} > 0$. For $N = \{\, i \mid \sum_j c_{ij} > 0 \,\}$, let $y_i = \sum_j (c_{ij}/\sum_h c_{ih}) x_j$ for all $i \in N$. Then $p_i \cdot (y_i - x_i) \leq 0$ for all $i \in N$ so $\Gamma(y_i,x_i) \leq 0$ for all $i \in N$.

For $i \in N$ if $y_i = x_i$, then $\Gamma(y_i,x_i) = 0$ and there are j and k with $j \neq k$ such that $c_{ij}, c_{ik} > 0$ because $c_{ii} = 0$. Therefore, if $y_i = x_i$, then strict concavity of Γ implies that $\sum_j (c_{ij}/(\sum_h c_{ih}))\Gamma(x_j,x_i) < \Gamma(y_i,x_i) = 0$. For $i \in N$ if $y_i \neq x_i$, then $\Gamma(y_i,x_i) < 0$ because if $\Gamma(y_i,x_i) = 0$, then strict concavity of Γ implies that $\Gamma((1-\tau)y_i + \tau x_i, x_i) > 0$ for all $\tau \in]0,1[$ contradicting that Γ represents $\mathscr{P}$ or $\mathscr{P}$ rationalizes $\mathscr{D}$. Therefore, if $y_i \neq x_i$, then strict concavity of Γ implies that $\sum_j (c_{ij}/(\sum_h c_{ih}))\Gamma(x_j,x_i) \leq \Gamma(y_i,x_i) < 0$. Hence $\sum_j c_{ij}\Gamma(x_j,x_i) < 0$ for all $i \in N$ so $\sum_{i,j} c_{ij}\Gamma(x_j,x_i) < 0$. However $\sum_{i,j} c_{ij}\Gamma(x_i,x_j) = 0$ because $c_{ji} = c_{ij}$ and $\Gamma(x_j,x_i) = -\Gamma(x_i,x_j)$. □

Lemma 2 *For a data set $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$ suppose that for all $c_{11},c_{12},\ldots,c_{\ell\ell} \geq 0$ with $c_{ii} = 0$ for all i and $c_{ji} = c_{ji}$ for all i and j,*

$$\sum_j c_{ij} p_i \cdot (x_j - x_i) \ \leq \ 0$$

for all i imply that $c_{ij} = 0$ for all i and j. Then $\mathscr{D}$ satisfies WLD.

Proof: For all i and j with $i < j$, let $v^{ij} \in \mathbb{R}^n$ be defined by

$$v_h^{ij} = \begin{cases} p_i \cdot (x_j - x_i) & \text{for } h = i \\ p_j \cdot (x_i - x_j) & \text{for } h = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $V \subset \mathbb{R}^n$ be the set of convex combinations of the vectors $(v^{ij})_{i<j}$

$$V = \left\{ v \in \mathbb{R}^n \mid \exists d \in \triangle^{(n-1)n/2} : v = \sum_{i<j} d_{ij} v^{ij} \right\}.$$

Then V is compact and $V \cap (-\mathbb{R}^n_+) = \emptyset$ because

$$\sum_{i<j} d_{ij} v^{ij} = \begin{pmatrix} \sum_{j=2}^{\ell} d_{1j} p_1 \cdot (x_j - x_1) \\ \vdots \\ \sum_{j=1}^{i-1} d_{ji} p_i \cdot (x_j - x_i) + \sum_{j=i+1}^{\ell} d_{ij} p_i \cdot (x_j - x_i) \\ \vdots \\ \sum_{j=1}^{\ell-1} d_{j\ell} p_\ell \cdot (x_j - x_\ell) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{\ell} c_{1j} p_1 \cdot (x_j - x_1) \\ \vdots \\ \sum_{j=1}^{\ell} c_{ij} p_i \cdot (x_j - x_i) \\ \vdots \\ \sum_{j+1}^{\ell} c_{\ell j} p_\ell \cdot (x_j - x_\ell) \end{pmatrix}$$

where $c_{ij} = d_{ij}$ for all i and j with $i < j$, $c_{ii} = 0$ for all i and $c_{ij} = d_{ji}$ for all i and j with $i > j$. For

$$\triangle^n_\varepsilon = \{ v \in \mathbb{R}^n \mid v_1 + \ldots + v_n = 1 \text{ and } v_i \geq -\varepsilon \text{ for all } i \}$$

let $C_\varepsilon \subset \mathbb{R}^n$ be the cone generated by $\triangle^n_\varepsilon$

$$C_\varepsilon = \{ v \in \mathbb{R}^n \mid \exists v' \in \triangle^n_\varepsilon, \delta \geq 0 : v = -\delta v' \}.$$

Then C_ε is closed and there is $\varepsilon > 0$ such that $V \cap C_\varepsilon = \emptyset$. Therefore there is $\lambda \in \mathbb{R}^n$ with $\lambda \neq 0$ such that $\sup_{v \in C_\varepsilon} \lambda \cdot v < \min_{v \in V} \lambda \cdot v$. If $\lambda_i \leq 0$ for some i, then $\sup_{v \in C_\varepsilon} \lambda \cdot v = \infty$. Hence $\lambda \in \mathbb{R}^n_{++}$ with $\sup_{v \in C_\varepsilon} \lambda \cdot v = 0$ so $\min_{v \in V} \lambda \cdot v > 0$. Thus $\lambda \cdot v^{ij} > 0$ for all v^{ij} and $\lambda \cdot v^{ij} = -(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j)$ for all i and j with $i < j$. □

'WLD implies Rationalizability'

The proof consists of three parts. Firstly an original data set $\mathscr{D} = \{(p_1, x_1), \ldots, (p_n, x_n)\}$ satisfying WLD is changed to a modified data set $\{(q_1, x_1), \ldots, (q_n, x_n)\}$. Secondly in Lemmas 3-5 it is shown that the modified data set $\{(q_1, x_1), \ldots, (q_n, x_n)\}$ can be rationalized by a preference relation that can be represented by a continuous function $k : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ that is monotone and concave in the first argument. Our proofs of Lemmas 3-5 are quite similar to part of the proof of Theorem 2 in John (2001) even though our proofs were made before we became aware of John (2001). Thirdly in Lemmas 6-7 it is shown that the function k can be modified to a function $\Gamma : \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \to \mathbb{R}$ that represents a smooth nontransitive preference relation that rationalizes the original data set $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$.

Let $u : \mathbb{R}^\ell_{++} \to \mathbb{R}$ be defined by

$$u(x) \;=\; \ln(x^1)+\ldots+\ln(x^\ell).$$

Suppose that the data set satisfies WLD for $\lambda_1,\ldots,\lambda_n > 0$. If $\varepsilon > 0$ is sufficiently small, then for all i, $q_i = p_i - (\varepsilon/\lambda_i)Du(x_i) \in \mathbb{R}^\ell_{++}$ and for all i and j with $i \neq j$,

$$(\lambda_i q_i - \lambda_j q_j)\cdot(x_i - x_j) \;<\; 0.$$

Let $\triangle^n \subseteq \mathbb{R}^n$ be the unit simplex

$$\triangle^n \;=\; \{v \in \mathbb{R}^n_+ \mid v_1+\ldots+v_n = 1\}.$$

Lemma 3 *Let $\mathscr{D} = \{(p_1,x_1),\ldots,(p_n,x_n)\}$ be a data set. Then the following are equivalent:*

(1) There are $\lambda_1,\ldots,\lambda_n > 0$ such that for all $i, j \in \{1,\ldots,n\}$, $i \neq j$ implies

$$(\lambda_i q_i - \lambda_j q_j)\cdot(x_i - x_j) \;<\; 0.$$

(2) There are a_{ij}, $i, j \in \{1,\ldots,n\}$, and b_i, $i \in \{1,\ldots,n\}$, with $a_{ji} = -a_{ij}$ for all i and j and $b_i > 0$ for all i, such that for all i and j with $i \neq j$

$$a_{ij} + (b_i q_i - b_j q_j)\cdot x_j \;>\; 0.$$

Proof: (1) $\Rightarrow$ (2): Letting

$$a_{ij} \;=\; -\frac{1}{2}(\lambda_i q_i - \lambda_j q_j)\cdot(x_i + x_j)$$

for all i and j and $b_i = \lambda_i$ for all i imply $a_{ji} = -a_{ij}$ and

$$a_{ij} + (b_i q_i - b_j q_j)\cdot x_j \;=\; -\frac{1}{2}(\lambda_i q_i - \lambda_j q_j)\cdot(x_i - x_j)$$

for all i and j. Therefore the inequalities in (2) are satisfied.

(2) $\Rightarrow$ (1): Adding the two inequalities for i and j with $i \neq j$ in (2) gives

$$(b_i q_i - b_j q_j)\cdot(x_i - x_j) \;<\; 0$$

Hence $(\lambda_1,\ldots,\lambda_n)$ with $\lambda_i = b_i$ for all i satisfies the inequalities in (1). □

Lemma 4 *Suppose that a_{ij}, $i,j \in \{1,\ldots,n\}$, and b_i, $i \in \{1,\ldots,n\}$, satisfy the inequalities in (2) in Lemma 3. Let the $(n \times n)$-matrix $M(y,z) = (m_{ij}(y,z))_{i,j=1}^n$ be defined by*

$$m_{ij}(y,z) = a_{ij} + b_i q_i \cdot y - b_j q_j \cdot z,$$

and the function $k : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ by

$$k(y,z) \ = \ \min_{\sigma \in \triangle^n} \max_{\tau \in \triangle^n} \sigma^T M(y,z) \tau.$$

Then for all y and z

(1) $M(z,y) = -M(y,z)^T$.

(2) k is continuous at (y,z).

(3) $k(z,y) = -k(y,z)$.

(4) $y \in \{z\} + \mathbb{R}_+ \backslash \{0\}^\ell$ implies $k(y,z) > 0$.

(5) $k(\cdot,z)$ is concave and $k(y,\cdot)$ is convex.

Proof: (1) follows from $a_{ji} = -a_{ij}$, $i,j \in \{1,\ldots,n\}$, because

$$\begin{aligned} m_{ji}(z,y) \ &= \ a_{ji} + b_j q_j \cdot z - b_i q_i \cdot y \\ &= \ -(a_{ij} + b_i q_i \cdot y - b_j q_j \cdot z) \ = \ -m_{ij}(y,z). \end{aligned}$$

(2) is a direct consequence of the definition of k.

For (3), it should be noted that $k(y,z)$ is the value of the two-person zero-sum (matrix) game with matrix $M(z,y)$, and the value of a matrix game is continuous w.r.t. the matrix defining the game. Thus the minimax theorem for matrix games (see e.g. Gale (1960)), which states that

$$\max_\tau \min_\sigma \sigma^T M(y,z) \tau \ = \ \min_\sigma \max_\tau \sigma^T M(y,z) \tau,$$

can be applied. Therefore

$$\begin{aligned} k(z,y) \ &= \ \min_\sigma \max_\tau \sigma^T M(z,y) \tau \ = \ \max_\tau \min_\sigma \sigma^T M(y,z) \tau = \max_\tau \min_\sigma -\sigma^T M(y,z)^T \tau \\ &= \ -\min_\tau \max_\sigma \sigma^T M(y,z)^T \tau \ = \ -\min_\tau \max_\sigma \tau^T M(y,z) \sigma \ = \ -k(y,z). \end{aligned}$$

For (4) and (5), let $\tau \in \triangle^n$. Then

$$M(y,z)\tau \ = \ \begin{pmatrix} \sum_j \left(a_{1j} - b_j q_j \cdot z\right) \tau_j + b_1 q_1 \cdot y \\ \vdots \\ \sum_j \left(a_{nj} - b_j q_j \cdot z\right) \tau_j + b_n q_n \cdot y \end{pmatrix},$$

so the set

$$\{\, \tau \in \triangle^n \mid \forall \tau' \in \triangle^n : \min_\sigma \sigma^T M(y,z)\tau \geq \min_\sigma \sigma^T M(y,z)\tau' \,\}$$

is independent of y. Hence if

$$k(y,z) \;=\; \min_\sigma \max_\tau \sigma^T M(y,z)\tau \;=\; \max_\tau \min_\sigma \sigma^T M(y,z)\tau \;=\; \min_\sigma \sigma^T M(y,z)\tau^0$$

for some $\tau^0 \in \triangle^n$, then $k(y',z) = \min_\sigma \sigma^T M(y',z)\tau^0$ for all y'.

For (4), there is $\tau^0 \in \triangle^n$ such that $M(z,z)\tau^0 \in \mathbb{R}^n_+$ because $k(z,z) = 0$ according to (3). For all i and j, if $y \in \{z\} + \mathbb{R}^\ell_+ \backslash \{0\}$, then $m_{ij}(y,z) > m_{ij}(z,z)$, because $m_{ij}(y,z) = m_{ij}(z,z) + b_i q_i(y-z)$. Thus $M(y,z)\tau^0 \in \mathbb{R}^n_{++}$, so $k(y,z) > 0$.

For (5), there is $\tau^0 \in \triangle^n$ such that for all y

$$k(y,z) \;=\; \min_i \left\{ \sum_j \left(a_{ij} - b_j q_j \cdot z\right) \tau^0_j + b_i q_i \cdot y \right\}.$$

Therefore $k(\cdot,z)$ is convex and $k(y,\cdot)$ is concave, because $k(y,\cdot) = -k(\cdot,y)$. □

Lemma 5 *Suppose that a_{ij}, $i,j \in \{1,\ldots,n\}$, and b_i, $i \in \{1,\ldots,n\}$, satisfy the inequalities in (2) in Lemma 3. Let $k : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ be defined as in Lemma 4. Then*

(6) For all i and $y \in \mathbb{R}^\ell_+$, $q_i \cdot (y - x_i) \leq 0$ implies $k(y,x_i) \leq 0$.

Proof: Since $a_{ji} = -a_{ij}$ it follows that

$$\begin{aligned} m_{ij}(y,x_i) &= a_{ij} + b_i q_i \cdot y - b_j q_j \cdot x_i \\ &= b_i q_i \cdot (y - x_i) + (a_{ij} + (b_i q_i - b_j q_j) \cdot x_i) \\ &= b_i q_i \cdot (y - x_i) - (a_{ji} + (b_j q_j - b_i q_i) \cdot x_i). \end{aligned}$$

Hence if $q_i \cdot (y - x_i) \leq 0$, then $m_{ij}(y,x_i) \leq 0$ for all j, so

$$k(y,x_i) \;\leq\; \max_j \left\{ m_{ij}(y,x_i) \right\}.$$

□

Lemma 6 *Suppose that a_{ij}, $i,j \in \{1,\ldots,n\}$, and b_i, $i \in \{1,\ldots,n\}$, satisfy the inequalities in (2) in Lemma 3. Then there is a smooth function $K : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ with $D_1 K(x_i,x_i) = b_i q_i$, $i \in \{1,\ldots,n\}$, satisfying (2)-(5) in Lemma 4 and (6) in Lemma 5.*

Proof: The inequalities in (2) in Lemma 3 imply that $m_{ij}(x_j,x_j) > 0$ for $i \neq j$, so $m_{ji}(x_j,x_j) < 0$ for $i \neq j$. Thus for $\tau^0 \in \triangle^n$ with $\tau_i^0 = 1$ for $i = j$ and $\tau_i^0 = 0$ for $i \neq j$ and $\tau \in \triangle^n$ with $\tau \neq \tau^0$

$$\begin{aligned} a_{jj} + (b_j q_j - b_j q_j) x_j &= \min_i \left\{ \sum_j \left(a_{ij} - b_j q_j \cdot x_j \right) \tau_j^0 + b_i q_i \cdot x_j \right\} \\ &> \min_i \left\{ \sum_j \left(a_{ij} - b_j q_j \cdot x_j \right) \tau_j + b_i q_i \cdot x_j \right\}. \end{aligned}$$

Therefore for every j there exists $\delta_j > 0$ such that if $\|(y,z) - (x_j,x_j)\| \leq \delta_j$, then

$$\begin{aligned} a_{jj} + b_j q_j \cdot y - b_j q_j \cdot z &= \min_i \left\{ \sum_j \left(a_{ij} - b_j q_j \cdot z \right) \tau_j^0 + b_i q_i \cdot y \right\} \\ &> \min_i \left\{ \sum_j \left(a_{ij} - b_j q_j \cdot z \right) \tau_j + b_i q_i \cdot y \right\}. \end{aligned}$$

For $\delta = \min_i\{\delta_i/3\}$ let $\rho : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}_+$ be smooth with: $\rho(v_y,v_z) > 0$ for $\|(v_y,v_z)\| < \delta$; $\rho(v_y,v_z) = 0$ for $\|(v_y,v_z)\| \geq 3\delta$; $\rho(-v_y,-v_z) = \rho(v_y,v_z)$; $\rho(v_z,v_y) = \rho(v_y,v_z)$; and, $\int \rho(v_y,v_z) d(v_y,v_z) = 1$.

There is a continuous function $k : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ satisfying (2)-(5) in Lemma 4 and (6) in Lemma 5. Let $K : \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}$ be the convolution of k and ρ,

$$K(y,z) = \int k(y - v_y, z - v_z)\, \rho(v_y,v_z)\, d(v_y,v_z).$$

Then K is smooth according to Theorem 2.3 on p. 227 in Lang (1993).

For (2), K is smooth.

For (3),

$$\begin{aligned} K(z,y) &= \int k(z - v_y, y - v_z)\, \rho(v_y,v_z)\, d(v_y,v_z) \\ &= -\int k(y - v_z, z - v_y)\, \rho(v_y,v_z)\, d(v_y,v_z) \\ &= -\int k(y - v_z, z - v_y)\, \rho(v_z,v_y)\, d(v_z,v_y) \\ &= -K(y,z). \end{aligned}$$

For (4), if $y \in \{z\} + \mathbb{R}^\ell_+ \backslash \{0\}$, then

$$\begin{aligned} K(y,z) &= \int k(y - v_y, z - v_z)\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &> \int k(z - v_y, z - v_z)\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &= K(z,z) \end{aligned}$$

and $K(z,z) = 0$ according to (3). Hence $K(y,z) > 0$.

For (5),

$$\begin{aligned} K((1-t)y + ty', z) &= \int k((1-t)y + ty' - v_y, z - v_z)\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &\geq \int [(1-t)k(y - v_y, z - v_z) + tk(y' - v_y, z - v_z)]\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &= (1-t)K(y,z) + tK(y', z). \end{aligned}$$

Thus $K(\cdot, z)$ is concave and $K(y, \cdot) = -K(\cdot, y)$, so $K(y, \cdot)$ is convex.

For (6), if $\|(y,z) - (x_i, x_i)\| \leq \delta$,

$$\begin{aligned} K(y,z) &= \int k(y - v_y, z_i - v_y)\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &= \int (a_{ii} + b_i q_i \cdot (y - v_y) - b_i q_i \cdot (z - v_z))\, \rho(v_y, v_z)\, d(v_y, v_z) \\ &= b_i q_i \cdot (y - z). \end{aligned}$$

Therefore, if $\|y - x_i\| \leq \delta$, then $q_i \cdot (y - x_i) \leq 0$ if and only if $K(y, x_i) \leq 0$. Suppose that there is y' with $q_i \cdot (y' - x_i) \leq 0$ and $K(y', x_i) > 0$. Then $K((1-t)y' + tx_i, x_i) > 0$ for all $t \in]0,1]$, because $K(\cdot, x_i)$ is concave according to (5). Hence there is $t \in]0,1]$ such that $\|((1-t)y + tx_i) - x_i\| \leq \delta$ and $K((1-t)y + tx_i, x_i) > 0$, but this contradicts that if $\|y - x_i\| \leq \delta$, then $q_i \cdot (y - x_i) \leq 0$ if and only if $K(y, x_i) \leq 0$.

Finally, it follows from the proof of (6) that $D_1 K(x_i, x_i) = b_i q_i$, $i \in \{1, \ldots, n\}$. □

Lemma 7 *Suppose that a_{ij}, $i, j \in \{1, \ldots, n\}$, and b_i, $i \in \{1, \ldots, n\}$, satisfy the inequalities in (2) in Lemma 3. Then $\Gamma : \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++} \to \mathbb{R}$ defined by*

$$\Gamma(y,z) \;=\; K(y,z) + \varepsilon(u(y) - u(z))$$

has the following properties:

(1) For all $y, z \in \mathbb{R}^\ell_{++}$, $\Gamma(z,y) = -\Gamma(y,z)$.

(2 Γ is smooth with $D_1\Gamma(x_i.x_i) = b_i p_i$ and $D_2\Gamma(x_i,x_i) = -b_i p_i$.

(3) For all $y,z \in \mathbb{R}^\ell_{++}$, $D_1\Gamma(y,z) \in \mathbb{R}^\ell_{++}$.

(4) For all $y,z \in \mathbb{R}^\ell_{++}$ and $v \in \mathbb{R}^\ell \setminus \{0\}$, $v^T D_{11}\Gamma(y,z)v < 0$.

(5) For all $z \in \mathbb{R}^\ell_{++}$, the set $\{\, y \mid \Gamma(y,z) \geq 0 \,\}$ is closed in $\mathbb{R}^\ell$.

Proof: For (1), $\Gamma(z,y) = -\Gamma(y,z)$ because $K(z,y) = -K(y,z)$ according to Lemma 6.

For (2), Γ is smooth because K is smooth according to Lemma 6 and u is smooth. Moreover $D_1\Gamma(x_i,x_i) = b_i q_i + \varepsilon Du(x_i) = b_i p_i - \varepsilon Du(x_i) + \varepsilon Du(x_i) = b_i p_i$. Thus $D_2\Gamma(x_i,x_i) = -D_1\Gamma(x_i,x_i) = -b_i p_i$, because $\Gamma(y,z) + \Gamma(z,y) = 0$.

For (3), $D_1\Gamma(y,z) \in \mathbb{R}^\ell_{++}$, because $K(\cdot,z)$ is increasing so $D_1K(y,z) \in \mathbb{R}^\ell_{+}$ and $Du(x) \in \mathbb{R}^\ell_{++}$ for all x.

For (4), $v^T D^2_{11}\Gamma(y,z)v < 0$, because $K(\cdot,z)$ is concave so $v^T D_{11}K(y,z)v \leq 0$ and $v^T D^2 u(x)v < 0$ for all x.

For (5), if $y,z \in \mathbb{R}^\ell_{++}$ and $\bar{y} \in \mathbb{R}^\ell_{+} \backslash \mathbb{R}^\ell_{++}$, $\lim_{y\to\bar{y}} \Gamma(y,z) = -\infty$ because $\lim_{y\to\bar{y}} u(y) = -\infty$. Therefore if $\Gamma(y,z) \geq 0$ and $y \to \bar{y}$, then $\bar{y} \in \mathbb{R}^\ell_{++}$ so $\{\, y \mid \Gamma(y,z) \geq 0 \,\}$ is closed in $\mathbb{R}^\ell_{++}$. □

Proof of 'WLD implies Rationalizability' in Theorem 2: Let the preference relation $\mathscr{P}$ be defined by $y\mathscr{P}z$ if and only if $\Gamma(y,z) \geq 0$ where Γ is defined in Lemma 7. Then $\mathscr{P}$ is a smooth nontransitive preference relation that rationalizes $\mathscr{D}$ according to Lemma 7. □

5 Rationalizability without the weighted law of demand

In the present section, convex, but not necessarily strictly convex, preference relations are considered. Therefore WARP and WLD have to be weakened. A data set $\mathscr{D}$ satisfies *WARP for correspondences* provided that for every pair of observations i and j, $p_i \cdot (x_i - x_j) \geq 0$ implies $p_j \cdot (x_j - x_i) \leq 0$. A data set $\mathscr{D}$ satisfies *WLD for correspondences* provided that there are $\lambda_1, \ldots, \lambda_n > 0$ such that for all i and j, $(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) \leq 0$. In Example 2 below a data set not satisfying WLD for correspondences but rationalizable by a preference relation satisfying WARP for correspondences, is presented. However the preference relation is weakly continuous rather than continuous where weakly continuous means that the set $\{y \mid x\mathscr{P}^\circ y\}$ is open for all x.

Example 1 above shows that data sets can satisfy WARP without being rationalizable by preference relations for which the associated demand function satisfies WARP. Demand functions satisfying WARP can be rationalized by preference relations satisfying completeness, strong monotonicity and weak forms of strict convexity and continuity as shown in Quah (2006). Therefore Example 1 implies that demand functions satisfying WARP also satisfy some other and stronger condition than WARP. Example 2 below shows that there are data sets not satisfying WLD, but still rationalizable by preference relations for which the associated demand correspondences satisfy WARP for correspondences. Hence WLD is not the stronger condition.

The preference relations used in Example 2 below are weakly continuous, complete, monotonic and convex.

Example 2: Consider a set of data consisting of four observations $\mathscr{D} = \{(p_1,x_1),\ldots,(p_4,x_4)\}$ where

$$
\begin{aligned}
p_1 &= (1,0,0,0) \qquad x_1 = \left(3, \frac{31}{10}, 5, \frac{28}{10}\right) \\
p_2 &= (0,1,0,0) \qquad x_2 = \left(\frac{28}{10}, 3, \frac{31}{10}, 5\right) \\
p_3 &= (0,0,1,0) \qquad x_3 = \left(5, \frac{28}{10}, 3, \frac{31}{10}\right) \\
p_4 &= (0,0,0,1) \qquad x_4 = \left(\frac{31}{10}, 5, \frac{28}{10}, 3\right).
\end{aligned}
$$

For every i

$$p_i \cdot x_i = 3,\; p_i \cdot x_{i+1} = \frac{28}{10},\; p_{i+1} \cdot x_i = \frac{31}{30}$$

with $i+1 = 1$ for $i = 4$. Therefore

$$(p_i - p_{i+1}) \cdot (x_i - x_{i+1}) = p_i \cdot (x_i - x_{i+1}) - p_{i+1} \cdot (x_i - x_{i+1}) = \frac{2}{10} - \frac{1}{10} = \frac{1}{10} > 0$$

for every i. Moreover, $p_i \cdot (x_i - x_j) < 0$ for all i and j with $j \notin \{i, i+1\}$, so the set of data satisfies WARP. By construction

$$\sum_{i=1}^{4} (\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) \;=\; \frac{1}{6} \sum_{i=1}^{4} \lambda_i.$$

Therefore there are no $\lambda_1,\ldots,\lambda_4 > 0$ such that $(\lambda_i p_i - \lambda_j p_j) \cdot (x_i - x_j) \leq 0$ for all i and j, so the set of data does not satisfy WLD for correspondences.

Next it is shown that there is a complete, monotonic and convex preference relation that rationalizes the set of data. A set of vectors $U(x)$ is assigned to every $x \in \mathbb{R}^4_+$ such that the preferred set at x is contained in the set $\cap_{p \in U(x)} \{z \mid p \cdot (z - x) \geq 0\}$. The construction is done in a sequence of steps.

Step 1: We separate off several sets that do not contain any of the points $x_1,\ldots,x_4$.

For $q_0 = (1,1,1,1)$ let

$$A_0 \;=\; \left\{x \in \mathbb{R}^4_+ \mid q_0 \cdot x < \frac{139}{10}\right\}.$$

and $T(x) = q_0$ for all $x \in A_0$. Then $q_0 \cdot (y - x) > 0$ for $x \in A_0$ and $y \notin A_0$.

For $q_1 = (0,2,3,41/10)$ let

$$A_1 \;=\; \left\{x \in \mathbb{R}^4_+ \backslash A_0 \mid q_1 \cdot x < \frac{273}{10}\right\},$$

and $T(x) = q_1$ on A_1. Then for $H_i = \left\{x \in \mathbb{R}^4_+ \mid p_i \cdot (x - x_i) \leq 0\right\}$ for all i, $(H_2 \cap H_3 \cap H_4) \backslash A_0 \subset A_1$, $x_i \notin A_1$ for all i, and if $x \in A_1$ and $y \notin A_0 \cup A_1$, then $q_1 \cdot (y - x) \geq 0$.

For $q_2 = (41/10,0,2,3)$ let

$$A_2 \;=\; \left\{x \in \mathbb{R}^4_+ \backslash (A_0 \cup A_1) \mid q_2 \cdot x < \frac{273}{10}\right\},$$

and $T(x) = q_2$ on A_2. Then $(H_1 \cap H_3 \cap H_4) \backslash (A_0 \cup A_1) \subset A_2$, $x_i \notin A_2$, all i, and if $x \in A_2$ and $y \notin A_0 \cup A_1 \cup A_2$, then $q_2 \cdot (y - x) \geq 0$.

For $q_3 = (3,41/10,0,2)$ let

$$A_3 \;=\; \left\{x \in \mathbb{R}^4_+ \backslash (A_0 \cup A_1 \cup A_2) \mid q_3 \cdot x < \frac{273}{10}\right\},$$

and $T(x) = q_3$ on A_3. Then $(H_1 \cap H_2 \cap H_4) \backslash (A_0 \cup A_1 \cup A_2) \subset A_3$, $x_i \notin A_3$, all i, and if $x \in A_3$ and $y \notin A_0 \cup A_1 \cup A_2 \cup A_3$, then $q_3 \cdot (y - x) \geq 0$.

Next, for $q_4 = (2,3,41/10,0)$ let

$$A_4 \;=\; \left\{x \in \mathbb{R}^4_+ \backslash (A_0 \cup A_1 \cup A_2 \cup A_3) \mid q_4 \cdot x < \frac{273}{10}\right\},$$

and $T(x) = q_4$ on A_4. Then $(H_1 \cap H_2 \cap H_3) \backslash (A_0 \cup A_1 \cup A_2 \cup A_3) \subset A_4$, $x_i \notin A_4$, all i, and if $x \in A_4$ and $y \notin A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$, then $q_4 \cdot (y - x) \geq 0$.

Step 2: We separate off a set that contains all the points $x_1, \ldots, x_4$.

For $A = \cup_{i=0}^4 A_i$ let

$$B \;=\; \{\, x \in \mathbb{R}^4_+ \backslash A \mid \exists i \in \{1, \ldots, 4\} : p_i \cdot x \leq 3 \,\}.$$

For $i, j \in \{1, \ldots, 4\}$, j dominates i provided $p_i \cdot (x_j - x_i) < 0$. Clearly if $x_j \in H_i$, then j dominates i. Hence j dominates i if and only if $j = i + 1$.

For $x \in B$, let $D(x) \subset \{1, \ldots, 4\}$ be the set of indices i such that $p_i \cdot (x - x_i) \leq 0$ and there is no j with $p_j \cdot (x - x_j) < 0$ that dominates i. Clearly, for every $x \in B$, $D(x)$ contains at least one element and at most two elements. Therefore for $x \in B$ let

$$T(x) \;=\; \{\, p_i \mid i \in D(x) \text{ and } \forall j \in D(x), p_i \cdot x \leq p_j \cdot x \,\}.$$

Step 3: We separate off the complement to $A \cup B$.

For $C = \mathbb{R}^4_+ \backslash (A \cup B)$, let $T(x) = q_0$ for all $x \in C$. Next, let the correspondence $U : \mathbb{R}^4_+ \rightrightarrows \{p_1, \ldots, p_4, q_0, q_1, \ldots, q_4\}$ be defined by

$$\operatorname{Graph} U \;=\; \operatorname{cl} T.$$

Then the correspondence U has closed graph by construction.

Step 4: We construct a preference relation $\mathscr{P}$.

Let the preference relation $\mathscr{P}$ be defined by

$$\mathscr{P}(x) \;=\; \{\, y \in \mathbb{R}^4_+ \mid p \cdot (y - x) \geq 0, p \in U(x)\,\}$$

where $\mathscr{P}(x) = \{\, y \in \mathbb{R}^4_+ \mid y \,\mathscr{P}\, x\,\}$. By construction, $\mathscr{P}(x)$ is convex and closed with nonempty interior for every $x \in \mathbb{R}^4_+$. Moreover, $\mathscr{P}$ is weakly continuous in the sense that for all $y \in \mathbb{R}^4_+$, the set

$$\mathscr{P}^{\circ -1}(y) \;=\; \{\, x \mid y \,\mathscr{P}^\circ\, x\,\}$$

is open. Indeed for all $x, y \in \mathbb{R}^4_+$ with $y \in \mathscr{P}^\circ(x)$ there is an open neighborhood G_x of x such that $U(z) \subseteq U(x)$ for all $z \in G_x$, so $y \in \mathscr{P}^\circ(z)$ for all $z \in G_x$.

Step 5: We show that the preference relation is complete.

For $x', y' \in \mathbb{R}^4_+$ assume that $y' \notin \mathscr{P}(x')$ and $x \notin \mathscr{P}(y')$. Then there are $p_{x'} \in U(x')$ and $p_{y'} \in U(y')$ such that $p_{x'} \cdot y' < p_{x'} \cdot x'$ and $p_{y'} \cdot x' < p_{y'} \cdot y'$. Therefore in every neighborhood of x' and in every neighborhood of y', there are x and y and $p_x \in T(x)$ and $p_y inT(y)$, such that $y \notin \mathscr{P}(x)$ and $x \notin \mathscr{P}(y)$ and $p_x \cdot y < p_x \cdot x$ and $p_y \cdot x < p_y \cdot y$.

If $x \in A^i$, $i = 0, 1, \ldots, 4$, then $p_x \cdot (y - x) \geq 0$ for $y \notin \cup_{j=0}^{i} A^j$. If $y \in C$, then $p_x \cdot (y - x) \geq 0$ for $x \notin C$ and $p_y = p_x$ for $x \in C$. Hence only the case where $x, y \in B$ needs to be considered.

Suppose that $x, y \in B$. Then there must be $p_i \in T(x)$ and $p_j \in T(y)$ such that

$$p_i \cdot y \;<\; p_i \cdot x \;<\; p_i \cdot x_i \;=\; 3 \text{ and } p_j \cdot x \;<\; p_j \cdot y \;<\; p_j \cdot x_j \;=\; 3.$$

Thus $i \neq j$ and $x, y \in H_i \cap H_j$. By the construction of B, no $x \in B$ can belong to more than two sets H_k for $k \in \{1, \ldots, 4\}$, so $D(x) = D(y) = \{i\}$ or $D(x) = D(y) = \{j\}$. This contradicts that $i \neq j$, so $y \in \mathscr{P}(x)$ or $x \in \mathscr{P}(y)$.

It has been shown that $\mathscr{P}$ is complete, convex and weakly continuous, but it remains to show that $\mathscr{P}$ is monotonic. Clearly

$$y \,\mathscr{P}^\circ\, x \iff p \cdot y \;>\; p \cdot x$$

for all $p \in T(x)$. Therefore $\{x\} + \mathbb{R}^4_{++} \subset \mathscr{P}(x)$ for all $x \in \mathbb{R}^4_+$, so $\mathscr{P}$ is monotonic. Finally, x_i is in the closure of A_i for every i, so $y \,\mathscr{P}^\circ\, x$ implies that $p_i \cdot y > p_i \cdot x$. Hence $\mathscr{P}$ rationalizes $\mathscr{D}$. *End of example*

6 Equilibrium behaviour of pure-exchange economies

In the present section, we introduce market data sets and apply Theorem 2 to characterize market data sets consistent with equilibrium behaviour of some pure-exchange economy with smooth nontransitive consumers.

Rationalizability of market data sets

A market data set $\mathscr{MD}$ consists of n observations of price vectors, lists of individual incomes for m consumers and aggregate demands,

$$\mathscr{MD} = \{(p_1, (w_1^h)_{h=1}^m, r_1), \ldots, (p_n, (w_n^h)_{h=1}^m, r_n)\}$$

where $p_i \in \mathbb{R}^\ell_{++}$, $r_i \in \mathbb{R}^\ell_{++}$ and $w_i^1 + \ldots + w_i^m = p_i \cdot r_i$ for all i and $w_i^h > 0$ for all i and h.

For market data sets, lists of individual incomes and aggregate demands rather than individual consumption bundles are observed. Therefore rationalizability reduces to whether aggregate demand can be split into individual consumption bundles such that individual data sets consisting of pairs of price vectors and consumption bundles can be rationalized by smooth nontransitive preference relations.

Definition 4 *A list of individual preference relations* $(\mathscr{P}^h)_{h=1}^m$ **rationalizes** *a market data set* $\mathscr{MD}$ *provided that there are lists of individual consumption bundles* $(x_1^h)_{h=1}^m, \ldots, (x_n^h)_{h=1}^m$ *with* $x_i^h \in \mathbb{R}^\ell_{++}$ *and* $p_i \cdot x_i^h = w_i^h$ *for all i and h and* $\sum_{h=1}^m x_i^h = r_i$ *for all i such that for all h,* $\mathscr{P}^h$ *rationalizes the data set* $\mathscr{D}^h = \{(p_1, x_1^h), \ldots, (p_n, x_n^h)\}$.

Characterization of rationalizable market data sets

Market data sets rationalizable by lists of individual smooth nontransitive preference relations are characterized in the following corollary to Theorem 2.

Corollary 1 *A market data set* $\mathscr{MD}$ *can be rationalized by a list of individual smooth nontransitive preference relations* $(\mathscr{P}^h)_{h=1}^m$ *if and only if there are lists of individual consumption bundles and weights* $(x_1^h, \lambda_1^h)_{h=1}^m, \ldots, (x_n^h, \lambda_n^h)_{h=1}^m$ *with* $(x_i^h, \lambda_i^h) \in \mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$ *and* $p_i \cdot x_i^h = w_i^h$ *for all i and h and* $\sum_h x_i^h = r_i$ *for all i such that for all i, j and h,*

$$\begin{cases} x_i^h = x_j^h & \text{for } (p_i, w_i^h) \text{ and } (p_j, w_j^h) \text{ collinear} \\ (\lambda_i^h p_i - \lambda_j^h p_j) \cdot (x_i^h - x_j^h) < 0 & \text{for } (p_i, w_i^h) \text{ and } (p_j, w_j^h) \text{ not collinear.} \end{cases}$$

Remark: Corollary 1 is analogous to Theorem 2 in Brown & Matzkin (1996) in which monotonic, concave and continuous utility functions rather than smooth nontransitive preference relations are considered.

7 Concluding remarks

In the present paper, rationalizability of data sets by smooth nontransitive preference relations is been studied. The main result is that data sets can be rationalized by a smooth nontransitive preference relation if and only if the data sets satisfy WLD. Moreover, the main result is used to characterize market data sets consistent with equilibrium behaviour of pure-exchange economies where consumers have smooth nontransitive preference relations.

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