Applications of Eigenvalues and Eigenvectors

22.2

Introduction

Many applications of matrices in both engineering and science utilize eigenvalues and, sometimes, eigenvectors. Control theory, vibration analysis, electric circuits, advanced dynamics and quantum mechanics are just a few of the application areas.

Many of the applications involve the use of eigenvalues and eigenvectors in the process of **transforming** a given matrix into a **diagonal** matrix and we discuss this process in this Section. We then go on to show how this process is invaluable in solving coupled differential equations of both first order and second order.

Prerequisites

Before starting this Section you should . . .

- have a knowledge of determinants and matrices
- have a knowledge of linear first order differential equations

Learning Outcomes

On completion you should be able to . . .

- diagonalize a matrix with distinct eigenvalues using the modal matrix
- solve systems of linear differential equations by the 'decoupling' method

1. Diagonalization of a matrix with distinct eigenvalues

Diagonalization means transforming a non-diagonal matrix into an equivalent matrix which is diagonal and hence is simpler to deal with.

A matrix A with distinct eigenvalues has, as we mentioned in Property 3 in HELM 22.1, eigenvectors which are linearly independent. If we form a matrix P whose columns are these eigenvectors, it can be shown that

$$\det(P) \neq 0$$

so that P^{-1} exists.

The product $P^{-1}AP$ is then a **diagonal** matrix D whose diagonal elements are the eigenvalues of A. Thus if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the distinct eigenvalues of A with associated eigenvectors $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ respectively, then

$$P = \left[\begin{array}{ccccccc} X^{(1)} & \vdots & X^{(2)} & \vdots & \cdots & \vdots & X^{(n)} \\ & & & & & & \end{array} \right]$$

will produce a product

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \\ 0 & \ldots & \ldots & \lambda_n \end{bmatrix}$$

We see that the order of the eigenvalues in D matches the order in which P is formed from the eigenvectors.

N.B.

(a) The matrix P is called the **modal matrix** of A

(b) Since D is a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are the same as those of A, then the matrices D and A are said to be **similar**.

(c) The transformation of A into D using

$$P^{-1}AP = D$$

is said to be a **similarity transformation.**

Example 8

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. Obtain the modal matrix P and calculate the product $P^{-1}AP$.

(The eigenvalues and eigenvectors of this particular matrix A were obtained earlier in this Workbook at page 7.)

Solution

The matrix A has two distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 5$ with corresponding eigenvectors $X_1 = \begin{bmatrix} x \\ -x \end{bmatrix}$ and $X_2 = \begin{bmatrix} x \\ x \end{bmatrix}$. We can therefore form the modal matrix from the simplest eigenvectors of these forms:

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(Other eigenvectors would be acceptable e.g. we could use $P = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$ but there is no reason to over complicate the calculation.)

It is easy to obtain the inverse of this 2×2 matrix P and the reader should confirm that:

$$P^{-1} = \frac{1}{\text{det}(P)} \text{adj}(P) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We can now construct the product $P^{-1}AP$:

$$\begin{aligned} P^{-1}AP &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 5 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

which is a diagonal matrix with entries the eigenvalues, as expected. Show (by repeating the method outlined above) that had we defined $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (i.e. interchanged the order in which the eigenvectors were taken) we would find $P^{-1}AP = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ (i.e. the resulting diagonal elements would also be interchanged.)

Task

The matrix $A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$ has eigenvalues -1 and 3 with respective eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If $P_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ write down the products $P_1^{-1}AP_1$, $P_2^{-1}AP_2$, $P_3^{-1}AP_3$

(You may not need to do detailed calculations.)

Your solution

Answer

$P_1^{-1}AP_1 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D_1 \qquad P_2^{-1}AP_2 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D_2 \qquad P_3^{-1}AP_3 = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D_3$

Note that $D_1 = D_2$, demonstrating that any eigenvectors of A can be used to form P. Note also that since the columns of P_1 have been interchanged in forming P_3 then so have the eigenvalues in D_3 as compared with D_1.

Matrix powers

If $P^{-1}AP = D$ then we can obtain A (i.e. make A the subject of this matrix equation) as follows:

Multiplying on the left by P and on the right by P^{-1} we obtain

$$PP^{-1}APP^{-1} = PDP^{-1}$$

Now using the fact that $PP^{-1} = P^{-1}P = I$ we obtain

$$IAI = PDP^{-1} \quad \text{and so}$$

$$A = PDP^{-1}$$

We can use this result to obtain the **powers** of a square matrix, a process which is sometimes useful in control theory. Note that

$$A^2 = A.A \qquad A^3 = A.A.A. \quad \text{etc.}$$

Clearly, obtaining high powers of A directly would in general involve many multiplications. The process is quite straightforward, however, for a diagonal matrix D, as this next Task shows.

Task

Obtain D^2 and D^3 if $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Write down D^{10}.

Your solution

Answer

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & (-2) \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & (-2)^3 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & -8 \end{bmatrix}$$

Continuing in this way: $D^{10} = \begin{bmatrix} 3^{10} & 0 \\ 0 & (-2)^{10} \end{bmatrix} = \begin{bmatrix} 58049 & 0 \\ 0 & 1024 \end{bmatrix}$

We now use the relation $A = PDP^{-1}$ to obtain a formula for powers of A in terms of the easily calculated powers of the diagonal matrix D:

$$A^2 = A.A = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

Similarly: $A^3 = A^2.A = (PD^2P^{-1})(PDP^{-1}) = PD^2(P^{-1}P)DP^{-1} = PD^3P^{-1}$

The general result is given in the following Key Point:

Key Point 2

For a matrix A with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and associated eigenvectors $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ then if

$$P = [X^{(1)} : X^{(2)} : \ \ldots \ : X^{(n)}]$$

$D = P^{-1}AP$ is a diagonal matrix such that

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad \text{and} \qquad A^k = PD^kP^{-1}$$

Example 9

If $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ find A^{23}. (Use the results of Example 8.)

Solution

We know from Example 8 that if $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ then $P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} = D$

where $P^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\therefore \quad A = PDP^{-1}$ and $A^{23} = PD^{23}P^{-1}$ using the general result in Key Point 2

i.e. $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5^{23} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

which is easily evaluated.

Exercise

Find a diagonalizing matrix P if

(a) $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}$

Verify, in each case, that $P^{-1}AP$ is diagonal, with the eigenvalues of A as its diagonal elements.

Answer

(a) $P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \qquad PAP^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}, \quad PAP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Systems of first order differential equations

Systems of first order ordinary differential equations arise in many areas of mathematics and engineering, for example in control theory and in the analysis of electrical circuits. In each case the basic unknowns are each a function of the time variable t. A number of techniques have been developed to solve such systems of equations; for example the Laplace transform. Here we shall use eigenvalues and eigenvectors to obtain the solution. Our first step will be to recast the system of ordinary differential equations in the **matrix form** $\dot{X} = AX$ where A is an $n \times n$ coefficient matrix of constants, X is the $n \times 1$ column vector of unknown functions and $\dot{X}$ is the $n \times 1$ column vector containing the **derivatives** of the unknowns.. The main step will be to use the modal matrix of A to diagonalise the system of differential equations. This process will transform $\dot{X} = AX$ into the form $\dot{Y} = DY$ where D is a **diagonal matrix**. We shall find that this new diagonal system of differential equations can be easily solved. This special solution will allow us to obtain the solution of the original system.

Task

Obtain the solutions of the pair of first order differential equations

$$\left.\begin{aligned} \dot{x} &= -2x \\ \dot{y} &= -5y \end{aligned}\right\} \qquad (1)$$

given the **initial conditions**

$$\begin{array}{lll} x(0) = 3 & \text{i.e. } x = 3 & \text{at } t = 0 \\ y(0) = 2 & \text{i.e. } y = 2 & \text{at } t = 0 \end{array}$$

(The notation is that $\dot{x} \equiv \dfrac{dx}{dt}, \quad \dot{y} \equiv \dfrac{dy}{dt}$)

[Hint: Recall, from your study of differential equations, that the general solution of the differential equation $\dfrac{dy}{dt} = Ky$ is $y = y_0e^{Kt}$.]

Your solution

Answer

Using the hint: $x = x_0e^{-2t}$ $\quad y = y_0e^{-5t}$ where $x_0 = x(0)$ and $y_0 = y(0)$.

From the given initial condition $x_0 = 3$ $\quad y_0 = 2$ so finally $x = 3e^{-2t}$ $\quad y = 2e^{-5t}$.

In the above Task although we had two differential equations to solve they were really quite separate. We needed no knowledge of matrix theory to solve them. However, we should note that the two differential equations can be written in matrix form.

Thus if $X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}$

the two equations (1) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e. $\dot{X} = AX$.

Task

Write in matrix form the pair of **coupled** differential equations

$$\left.\begin{aligned} \dot{x} &= 4x + 2y \\ \dot{y} &= -x + y \end{aligned}\right\} \qquad (2)$$

Your solution

Answer

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{X} \quad = \quad A\,X$$

The essential difference between the two pairs of differential equations just considered is that the pair (1) were really separate equations whereas pair (2) were coupled:

- The first equation of (1) involving only the unknown x, the second involving only y. In matrix terms this corresponded to a **diagonal** matrix A in the system $\dot{X} = AX$.
- The pair (2) were coupled in that **both** equations involved **both** x and y. This corresponded to the **non-diagonal** matrix A in the system $\dot{X} = AX$ which you found in the last Task.

Clearly the second system here is more difficult to deal with than the first and this is where we can use our knowledge of diagonalization.

Consider a system of differential equations written in matrix form: $\dot{X} = AX$ where

$$X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \dot{X} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix}$$

We now introduce a new column vector of unknowns $Y = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}$ through the relation

$$X = PY$$

where P is the modal matrix of A. Then, since P is a matrix of constants:

$$\dot{X} = P\dot{Y} \qquad \text{so} \quad \dot{X} = AX \quad \text{becomes} \quad P\dot{Y} = A(PY)$$

Then, multiplying by P^{-1} on the left, $\quad \dot{Y} = (P^{-1}AP)Y$

But, because of the properties of the modal matrix, we know that $P^{-1}AP$ is a **diagonal matrix**. Thus if λ_1, λ_2 are **distinct** eigenvalues of A then:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Hence $\dot{Y} = (P^{-1}AP)Y$ becomes

$$\begin{bmatrix} \dot{r} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

That is, when written out we have

$$\begin{aligned} \dot{r} &= \lambda_1 r \\ \dot{s} &= \lambda_2 s. \end{aligned}$$

These equations are **decoupled**. The first equation only involves the unknown function $r(t)$ and has solution $r(t) = Ce^{\lambda_1 t}$. The second equation only involves the unknown function $s(t)$ and has solution $s(t) = Ke^{\lambda_2 t}$. $[C, K$ are arbitrary constants.]

Once r, s are known the original unknowns x, y can be found from the relation $X = PY$.

Note that the theory outlined above is more widely applicable as specified in the next Key Point:

Key Point 3

For any system of differential equations of the form

$$\dot{X} = AX$$

where A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and t is the independent variable the solution is

$$X = PY$$

where P is the modal matrix of A and

$$Y = [C_1\mathrm{e}^{\lambda_1 t},\ C_2\mathrm{e}^{\lambda_2 t}, \ldots, C_n\mathrm{e}^{\lambda_n t}]^T$$

Example 10

Find the solution of the coupled differential equations

$$\dot{x} = 4x + 2y$$

$$\dot{y} = -x + y \qquad \text{with initial conditions} \qquad x(0) = 1 \qquad y(0) = 0$$

Here $\dot{x} \equiv \dfrac{dx}{dt}$ and $\dot{y} \equiv \dfrac{dy}{dt}$.

Solution

Here $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$. It is easily checked that A has distinct eigenvalues $\lambda_1 = 3$ $\lambda_2 = 2$ and corresponding eigenvectors $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Therefore, taking $P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ then $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

and using Key Point 3, $r(t) = Ce^{3t}$ $\quad s(t) = Ke^{2t}$.

So
$$\begin{bmatrix} x \\ y \end{bmatrix} \equiv X = PY = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} Ce^{3t} \\ Ke^{2t} \end{bmatrix} = \begin{bmatrix} -2Ce^{3t} + Ke^{2t} \\ Ce^{3t} - Ke^{2t} \end{bmatrix}.$$

Therefore $\quad x = -2Ce^{3t} + Ke^{2t}$ and $y = Ce^{3t} - Ke^{2t}$.

We can now impose the initial conditions $x(0) = 1$ and $y(0) = 0$ to give

$$\begin{aligned} 1 &= -2C + K \\ 0 &= C - K. \end{aligned}$$

Thus $C = K = -1$ and the solution to the original system of differential equations is

$$\begin{aligned} x(t) &= 2e^{3t} - e^{2t} \\ y(t) &= -e^{3t} + e^{2t} \end{aligned}$$

The approach we have demonstrated in Example 10 can be extended to

(a) Systems of first order differential equations with n unknowns (Key Point 3)

(b) Systems of second order differential equations (described in the next subsection).

The only restriction, as we have said, is that the matrix A in the system $\dot{X} = AX$ has distinct eigenvalues.

3. Systems of second order differential equations

The decoupling method discussed above can be readily extended to this situation which could arise, for example, in a mechanical system consisting of coupled springs.
A typical example of such a system with two unknowns has the form

$$\ddot{x} = ax + by \qquad\qquad \ddot{y} = cx + dy$$

or, in matrix form,

$$\ddot{X} = AX \quad \text{where} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}$$

Task

Make the substitution $X = PY$ where $Y = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}$ and P is the modal matrix of A, A being assumed here to have distinct eigenvalues λ_1 and λ_2. Solve the resulting pair of decoupled equations for the case, which arises in practice, where λ_1 and λ_2 are both negative.

Your solution

Answer

Exactly as with a first order system, putting $X = PY$ into the second order system $\ddot{X} = AX$ gives

$$\ddot{Y} = P^{-1}APY \quad \text{that is} \quad \ddot{Y} = DY \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } \ddot{Y} = \begin{bmatrix} \ddot{r} \\ \ddot{s} \end{bmatrix} \text{ so}$$

$$\begin{bmatrix} \ddot{r} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

That is, $\ddot{r} = \lambda_1 r = -\omega_1^2 r$ and $\ddot{s} = \lambda_2 s = -\omega_2^2 s$ (where λ_1 and λ_2 are both negative.)

The two decoupled equations are of the form of the differential equation governing simple harmonic motion. Hence the general solution is

$$r = K\cos\omega_1 t + L\sin\omega_1 t \quad \text{and} \quad s = M\cos\omega_2 t + N\sin\omega_2 t$$

The solutions for x and y are then obtained by use of $X = PY$.

Note that in this second order case four initial conditions, two each for both x and y, are required because four constants K, L, M, N arise in the solution.

Exercises

1. Solve by decoupling each of the following **first order** systems:

(a) $\dfrac{dX}{dt} = AX$ where $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(b) $\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_1 + 3x_3 \qquad \dot{x}_3 = x_2$ with $x_1(0) = 2, \; x_2(0) = 0, \; x_3(0) = 2$

(c) $\dfrac{dX}{dt} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} X$, with $X(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(d) $\dot{x}_1 = x_1 \qquad \dot{x}_2 = -2x_2 + x_3 \qquad \dot{x}_3 = 4x_2 + x_3$ with $x_1(0) = x_2(0) = x_3(0) = 1$

2. Matrix methods can be used to solve systems of **second order** differential equations such as might arise with coupled electrical or mechanical systems. For example the motion of two masses m_1 and m_2 vibrating on coupled springs, neglecting damping and spring masses, is governed by

$$\begin{aligned} m_1\ddot{y}_1 &= -k_1y_1 + k_2(y_2 - y_1) \\ m_2\ddot{y}_2 &= -k_2(y_2 - y_1) \end{aligned}$$

where dots denote derivatives with respect to time.

Write this system as a matrix equation $\ddot{Y} = AY$ and use the decoupling method to find Y if

(a) $m_1 = m_2 = 1, \; k_1 = 3, \; k_2 = 2$

and the initial conditions are $y_1(0) = 1$, $y_2(0) = 2$, $\dot{y}(0) = -2\sqrt{6}$, $\dot{y}_2(0) = \sqrt{6}$

(b) $m_1 = m_2 = 1, \quad k_1 = 6, \; k_2 = 4$

and the initial conditions are $y_1(0) = y_2(0) = 0, \quad \dot{y}_1(0) = \sqrt{2}, \quad \dot{y}_2(0) = 2\sqrt{2}$

Verify your solutions by substitution in each case.

Answers

1. (a) $X = \begin{bmatrix} 2e^{5t} & - & e^{-5t} \\ e^{5t} & + & 2e^{-5t} \end{bmatrix}$ (b) $X = \begin{bmatrix} 2 \; \cosh 2t \\ 4 \; \sinh 2t \\ 2 \; \cosh 2t \end{bmatrix}$

(c) $X = \dfrac{1}{4} \begin{bmatrix} e^{5t} & + & 3e^{t} \\ e^{5t} & - & e^{t} \\ e^{5t} & - & e^{t} \end{bmatrix}$ (d) $X = \dfrac{1}{5} \begin{bmatrix} 5e^{t} & & \\ 2e^{2t} & + & 3e^{-3t} \\ 8e^{2t} & - & 3e^{-3t} \end{bmatrix}$

2. (a) $Y = \begin{bmatrix} \cos t - 2\sin\sqrt{6}t \\ 2\cos t + \sin\sqrt{6}t \end{bmatrix}$ (b) $Y = \begin{bmatrix} \sin\sqrt{2}t \\ 2\sin\sqrt{2}t \end{bmatrix}$