Contents 6

Exponential and Logarithmic Functions

Learning outcomes

In this Workbook you will learn about one of the most important functions in mathematics, science and engineering - the exponential function. You will learn how to combine exponential functions to produce other important functions, the hyperbolic functions, which are related to the trigonometric functions.

You will also learn about logarithms and the logarithmic function which is the function inverse to the exponential function. Finally you will learn what a log-linear graph is and how it can be used to simplify the presentation of certain kinds of data.

The Exponential Function

6.1

Introduction

In this Section we revisit the use of exponents. We consider how the expression a^x is defined when a is a positive number and x is *irrational*. Previously we have only considered examples in which x is a *rational* number. We consider these exponential functions $f(x) = a^x$ in more depth and in particular consider the special case when the base a is the exponential constant e where :

$$\text{e} = 2.7182818\ldots$$

We then examine the behaviour of e^x as $x \to \infty$, called **exponential growth** and of e^{-x} as $x \to \infty$ called **exponential decay**.

Prerequisites

Before starting this Section you should ...

- have a good knowledge of indices and their laws
- have knowledge of rational and irrational numbers

Learning Outcomes

On completion you should be able to ...

- approximate a^x when x is irrational
- describe the behaviour of a^x: in particular the exponential function e^x
- understand the terms **exponential growth** and **exponential decay**

1. Exponents revisited

We have seen (in HELM 1.2) the meaning to be assigned to the expression a^p where a is a positive number. We remind the reader that 'a' is called the **base** and 'p' is called the **exponent** (or power or index). There are various cases to consider:

If m, n are positive integers

- $a^n = a \times a \times \cdots \times a$ with n terms
- $a^{1/n}$ means the n^{th} root of a. That is, $a^{1/n}$ is that positive number which satisfies

  $(a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n}) = a$

  where there are n terms on the left hand side.
- $a^{m/n} = (a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n})$ where there are m terms.
- $a^{-n} = \dfrac{1}{a^n}$

For convenience we again list the basic laws of exponents:

Key Point 1

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^1 = a, \qquad \text{and if} \quad a \neq 0 \qquad a^0 = 1$$

Example 1

Simplify the expression $\dfrac{p^{n-2}p^m}{p^3p^{2m}}$

Solution

First we simplify the numerator:

$$p^{n-2}p^m = p^{n-2+m}$$

Next we simplify the denominator:

$$p^3p^{2m} = p^{3+2m}$$

Now we combine them and simplify:

$$\frac{p^{n-2}p^m}{p^3p^{2m}} = \frac{p^{n-2+m}}{p^{3+2m}} = p^{n-2+m}p^{-3-2m} = p^{n-2+m-3-2m} = p^{n-m-5}$$

Task Simplify the expression $\dfrac{b^{m-n}b^3}{b^{2m}}$

First simplify the numerator:

Your solution

$b^{m-n}b^3 =$

Answer

$b^{m-n}b^3 = b^{m+3-n}$

Now include the denominator:

Your solution

$$\frac{b^{m-n}b^3}{b^{2m}} = \frac{b^{m+3-n}}{b^{2m}} =$$

Answer

$$\frac{b^{m+3-n}}{b^{2m}} = b^{m+3-n-2m} = b^{3-m-n}$$

Task Simplify the expression $\dfrac{(5a^m)^2a^2}{(a^3)^2}$

Simplify the numerator:

Your solution

$(5a^m)^2a^2 =$

Answer

$(5a^m)^2a^2 = 25a^{2m}a^2 = 25a^{2m+2}$

Now include the denominator:

Your solution

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} =$$

Answer

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} = 25a^{2m+2-6} = 25a^{2m-4}$$

a^x when x is any real number

So far we have given the meaning of a^p where p is an integer or a rational number, that is, one which can be written as a quotient of integers. So, if p is rational, then

$$p = \frac{m}{n} \qquad \text{where } m, n \text{ are integers}$$

Now consider x as a real number which cannot be written as a rational number. Two common examples of these **irrational** numbers are $\sqrt{2}$ and π. What we shall do is *approximate* x by a rational number by working to a fixed number of decimal places. For example if

$$x = 3.14159265\ldots$$

then, if we are working to 3 d.p. we would write

$$x \approx 3.142$$

and **this** number can certainly be expressed as a rational number:

$$x \approx 3.142 = \frac{3142}{1000}$$

so, in this case

$$a^x = a^{3.14159\ldots} \approx a^{3.142} = a^{\frac{3142}{1000}}$$

and the final term: $a^{\frac{3142}{1000}}$ can be determined in the usual way by calculator. Henceforth we shall therefore assume that the expression a^x is defined for all positive values of a and for **all** real values of x.

Task

By working to 3 d.p. find, using your calculator, the value of $3^{\pi/2}$.

First, approximate the value of $\frac{\pi}{2}$:

Your solution

$\frac{\pi}{2} \approx$ to 3 d.p.

Answer

$$\frac{\pi}{2} \approx \frac{3.1415927\ldots}{2} = 1.5707963\cdots \approx 1.571$$

Now determine $3^{\pi/2}$:

Your solution

$3^{\pi/2} \approx$

Answer

$3^{\pi/2} \approx 3^{1.571} = 5.618$ to 3 d.p.

2. Exponential functions

For a fixed value of the base a the expression a^x clearly varies with the value of x: it is a function of x. We show in Figure 1 the graphs of $(0.5)^x, (0.3)^x, 1^x, 2^x$ and 3^x.

The functions a^x (as different values are chosen for a) are called **exponential functions**. From the graphs we see (and these are true for *all* exponential functions):

If $a > b > 0$ then

$a^x > b^x$ if $x > 0$ and $a^x < b^x$ if $x < 0$

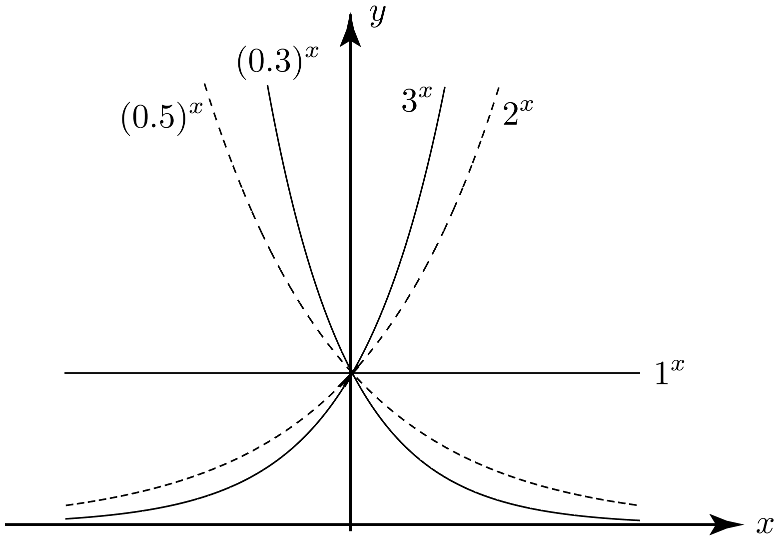

Figure 1: $y = a^x$ for various values of a

The most important and widely used exponential function has the particular base e $= 2.7182818\ldots$. It will not be clear to the reader why this particular value is so important. However, its importance will become clear as your knowledge of mathematics increases. The number e is as important as the number π and, like π, is also irrational. The approximate value of e is stored in most calculators. There are numerous ways of calculating the value of e. For example, it can be shown that the value of e is the end-point of the sequence of numbers:

$$\left(\frac{2}{1}\right)^1, \quad \left(\frac{3}{2}\right)^2, \quad \left(\frac{4}{3}\right)^3, \quad \ldots, \qquad \left(\frac{17}{16}\right)^{16}, \quad \ldots, \quad \left(\frac{65}{64}\right)^{64}, \ldots$$

which, in decimal form (each to 6 d.p.) are

$$2.000000, \quad 2.250000, \quad 2.370370, \quad \ldots, \quad 2.637929, \quad \ldots, 2.697345, \quad \ldots$$

This is a slowly converging sequence. However, it does lead to a precise definition for the value of e:

$$\mathrm{e} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

An quicker way of calculating e is to use the (infinite) series:

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \ldots$$

where, we remember,

$$n! = n \times (n-1) \times (n-2) \times \ldots (3) \times (2) \times (1)$$

(This is discussed more fully in HELM 16: Sequences and Series.)

Although all functions of the form a^x are called exponential functions we usually refer to e^x as *the* exponential function.

Key Point 2

e^x is **the** exponential function where $\mathrm{e} = 2.71828\ldots$

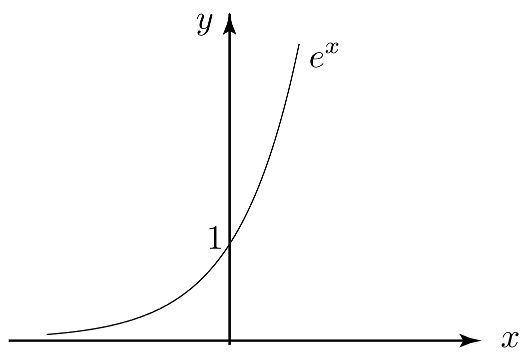

Figure 2: $y = \mathrm{e}^x$

Exponential functions (and variants) appear in various areas of mathematics and engineering. For example, the shape of a hanging chain or rope, under the effect of gravity, is well described by a combination of the exponential curves e^{kx}, e^{-kx}. The function e^{-x^2} plays a major role in statistics; it being fundamental in the important normal distribution which describes the variability in many naturally occurring phenomena. The exponential function e^{-kx} appears directly, again in the area of statistics, in the Poisson distribution which (amongst other things) is used to predict the number of events (which occur randomly) in a given time interval.

From now on, when we refer to an exponential function, it will be to the function e^x (Figure 2) that we mean, unless stated otherwise.

Task

Use a calculator to determine the following values correct to 2 d.p.
(a) $\mathrm{e}^{1.5}$, (b) e^{-2}, (c) e^{17}.

Your solution

(a) $\mathrm{e}^{1.5} =$ (b) $\mathrm{e}^{-2} =$ (c) $\mathrm{e}^{17} =$

Answer

(a) $\mathrm{e}^{1.5} = 4.48$, (b) $\mathrm{e}^{-2} = 0.14$, (c) $\mathrm{e}^{17} = 2.4 \times 10^7$

Task

Simplify the expression $\dfrac{\mathrm{e}^{2.7}\mathrm{e}^{-3(1.2)}}{\mathrm{e}^2}$ and determine its numerical value to 3 d.p.

First simplify the expression:

Your solution

$$\frac{\mathrm{e}^{2.7}\mathrm{e}^{-3(1.2)}}{\mathrm{e}^2} =$$

Answer

$$\frac{\mathrm{e}^{2.7}\mathrm{e}^{-3(1.2)}}{\mathrm{e}^2} = \mathrm{e}^{2.7}\mathrm{e}^{-3.6}\mathrm{e}^{-2} = \mathrm{e}^{2.7-3.6-2} = \mathrm{e}^{-2.9}$$

Now evaluate its value to 3 d.p.:

Your solution

$e^{-2.9} =$

Answer

0.055

3. Exponential growth

If $a > 1$ then it can be shown that, no matter how large K is:

$$\frac{a^x}{x^K} \to \infty \quad \text{as} \quad x \to \infty$$

That is, if K is fixed (though chosen as large as desired) then eventually, as x increases, a^x will become larger than the value x^K provided $a > 1$. The growth of a^x as x increases is called **exponential growth**.

Task

A function $f(x)$ grows exponentially and is such that $f(0) = 1$ and $f(2) = 4$. Find the exponential curve that fits through these points. Assume the function is $f(x) = \mathrm{e}^{kx}$ where k is to be determined from the given information. Find the value of k.

First, find $f(0)$ and $f(2)$ by substituting in $f(x) = \mathrm{e}^{kx}$:

Your solution

When $x = 0 \quad f(0) = \mathrm{e}^0 = 1$

When $x = 2, \quad f(2) = 4 \quad$ so $\mathrm{e}^{2k} = 4$

By trying values of k: $0.6, 0.7, 0.8$, find the value such that $\mathrm{e}^{2k} \approx 4$:

Your solution

$\mathrm{e}^{2(0.6)} =$ $\qquad$ $\mathrm{e}^{2(0.7)} =$ $\qquad$ $\mathrm{e}^{2(0.8)} =$

Answer

$\mathrm{e}^{2(0.6)} = 3.32$ (too low) $\qquad$ $\mathrm{e}^{2(0.7)} = 4.055$ (too high)

Now try values of k: $k = 0.67,\ 0.68,\ 0.69$:

Your solution

$\mathrm{e}^{2(0.67)} =$ $\qquad$ $\mathrm{e}^{2(0.68)} =$ $\qquad$ $\mathrm{e}^{2(0.69)} =$

Answer

$\mathrm{e}^{2(0.67)} = 3.819$ (low) $\qquad$ $\mathrm{e}^{2(0.68)} = 3.896$ (low) $\qquad$ $\mathrm{e}^{2(0.69)} = 3.975$ (low)

Next try values of $k = 0.691,\ 0.692$:

Your solution

$\mathrm{e}^{2(0.691)} =$ $\qquad$ $\mathrm{e}^{2(0.692)} =$ $\qquad$ $\mathrm{e}^{2(0.693)} =$

Answer

$\mathrm{e}^{2(0.691)} = 3.983,$ (low) $\qquad$ $\mathrm{e}^{2(0.692)} = 3.991$ (low) $\qquad$ $\mathrm{e}^{2(0.693)} = 3.999$ (low)

Finally, state the exponential function with k to 3 d.p. which most closely satisfies the conditions:

Your solution

$y =$

Answer

The exponential function is $\mathrm{e}^{0.693x}$.

We shall meet, in Section 6.4, a much more efficient way of finding the value of k.

4. Exponential decay

As we have noted, the behaviour of e^x as $x \to \infty$ is called exponential growth. In a similar manner we characterise the behaviour of the function e^{-x} as $x \to \infty$ as **exponential decay**. The graphs of e^x and e^{-x} are shown in Figure 3.

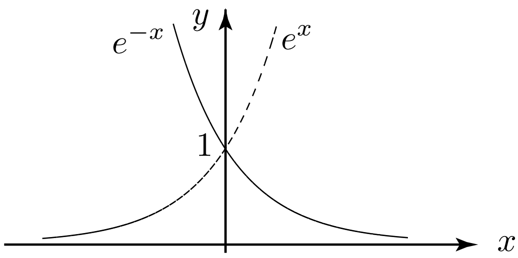

Figure 3: $y = e^x$ and $y = e^{-x}$

Exponential growth is very rapid and similarly exponential decay is also very rapid. In fact e^{-x} tends to zero so quickly as $x \to \infty$ that, no matter how large the constant K is,

$$x^K e^{-x} \to 0 \quad \text{as} \quad x \to \infty$$

The next Task investigates this.

Task

Choose $K = 10$ in the expression $x^K e^{-x}$ and calculate $x^K e^{-x}$ using your calculator for $x = 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35.$

Your solution

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$							

Answer

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$	6.6×10^4	4.5×10^5	1.7×10^5	2.1×10^4	1324	55	1.7

The topics of exponential growth and decay are explored further in Section 6.5.

Exercises

1. Find, to 3 d.p., the values of

   (a) 2^{-8} (b) $(5.1)^4$ (c) $(2/10)^{-3}$ (d) $(0.111)^6$ (e) $2^{1/2}$ (f) π^π (g) $e^{\pi/4}$ (h) $(1.71)^{-1.71}$

2. Plot $y = x^3$ and $y = e^x$ for $0 < x < 7$. For which integer values of x is $e^x > x^3$?

Answers

1. (a) 0.004 (b) 676.520 (c) 125 (d) 0.0 (e) 1.414 (f) 36.462 (g) 2.193 (h) 0.400
2. For integer values of x, $e^x > x^3$ if $x \geq 5$