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Matrices

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Learning outcomes

In this Workbook you will learn about matrices. In the first instance you will learn about the algebra of matrices: how they can be added, subtracted and multiplied. You will learn about a characteristic quantity associated with square matrices - the determinant. Using knowledge of determinants you will learn how to find the inverse of a matrix. Also, a second method for finding a matrix inverse will be outlined - the Gaussian elimination method.

A working knowledge of matrices is a vital attribute of any mathematician, engineer or scientist. You will find that matrices arise in many varied areas of science.

Introduction to Matrices

7.1

Introduction

When we wish to solve large systems of simultaneous linear equations, which arise for example in the problem of finding the forces on members of a large framed structure, we can isolate the coefficients of the variables as a block of numbers called a matrix. There are many other applications matrices. In this Section we develop the terminology and basic properties of a matrix.

Prerequisites

Before starting this Section you should ...

- be familiar with the rules of number algebra

Learning Outcomes

On completion you should be able to ...

- express a system of linear equations in matrix form
- recognise and use the basic terminology associated with matrices
- carry out addition and subtraction with two given matrices or state that the operation is not possible

1. Applications of matrices

The solution of simultaneous linear equations is a task frequently occurring in engineering. In electrical engineering the analysis of circuits provides a ready example.
However the simultaneous equations arise, we need to study two things:

(a) how we can conveniently represent large systems of linear equations

(b) how we might find the solution of such equations.

We shall discover that knowledge of the theory of matrices is an essential mathematical tool in this area.

Representing simultaneous linear equations

Suppose that we wish to solve the following three equations in three unknowns x_1, x_2 and x_3:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 3 \\ x_1 - x_2 + x_3 &= 4 \\ 2x_1 + 3x_2 + 4x_3 &= 5 \end{aligned}$$

We can isolate three facets of this system: the **coefficients** of x_1, x_2, x_3; the **unknowns** x_1, x_2, x_3; and the **numbers** on the right-hand sides.
Notice that in the system

$$\begin{aligned} 3x + 2y - z &= 3 \\ x - y + z &= 4 \\ 2x + 3y + 4z &= 5 \end{aligned}$$

the only difference from the first system is the names given to the unknowns. It can be checked that the first system has the solution $x_1 = 2,\ x_2 = -1,\ x_3 = 1$. The second system therefore has the solution $x = 2, y = -1, z = 1$.
We can isolate the three facets of the first system by using **arrays** of numbers and of unknowns:

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Even more conveniently we represent the arrays with letters (usually capital letters)

$$AX = B$$

Here, to be explicit, we write

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Here A is called the **matrix of coefficients**, X is called the **matrix of unknowns** and B is called the **matrix of constants**.
If we now append to A the column of right-hand sides we obtain the **augmented matrix** for the system:

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 3 \\ 1 & -1 & 1 & 4 \\ 2 & 3 & 4 & 5 \end{array}\right]$$

The order of the entries, or elements, is crucial. For example, all the entries in the second row relate to the second equation, the entries in column 1 are the coefficients of the unknown x_1, and those in the last column are the constants on the right-hand sides of the equations.
In particular, the entry in row 2 column 3 is the coefficient of x_3 in equation 2.

Representing networks

Shortest-distance problems are important in communications study. Figure 1 illustrates schematically a system of four towns connected by a set of roads.

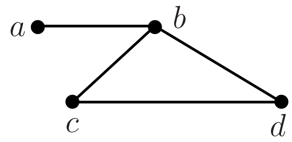

Figure 1

The system can be represented by the matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

The row refers to the town from which the road starts and the column refers to the town where the road ends. An entry of 1 indicates that two towns are directly connected by a road (for example b and d) and an entry of zero indicates that there is no direct road (for example a and c). Of course, if there is a road from b to d (say) it is also a road from d to b.

In this Section we shall develop some basic ideas about matrices.

2. Definitions

An array of numbers, rectangular in shape, is called a **matrix**. The first matrix below has 3 rows and 2 columns and is said to be a '3 by 2' matrix (written 3×2). The second matrix is a '2 by 4' matrix (written 2×4).

$$\left[\begin{array}{rr} 1 & 4 \\ -2 & 3 \\ 2 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{array}\right]$$

The general 3×3 matrix can be written

$$A = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

where a_{ij} denotes the element in row i, column j.
For example in the matrix:

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 6 & -12 \\ 5 & 7 & 123 \end{bmatrix}$$

$$a_{11} = 0, \quad a_{12} = -1, \quad a_{13} = -3, \quad \ldots \quad a_{22} = 6, \quad \ldots \quad a_{32} = 7, \quad a_{33} = 123$$

Key Point 1

The General Matrix

A general $m \times n$ matrix A has m rows and n columns.

The entries in the matrix A are called the **elements** of A.

In matrix A the element in row i and column j is denoted by a_{ij}.

A matrix with only one column is called a **column vector** (or **column matrix**).

For example, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ are both 3×1 column vectors.

A matrix with only one row is called a **row vector** (or **row matrix**). For example $[2, -3, 8, 9]$ is a 1×4 row vector. Often the entries in a row vector are separated by commas for clarity.

Square matrices

When the number of rows is the same as the number of columns, i.e. $m = n$, the matrix is said to be **square** and of **order** n (or m).

- In an $n \times n$ square matrix A, the **leading diagonal** (or **principal diagonal**) is the 'north-west to south-east' collection of elements $a_{11}, a_{22}, \ldots, a_{nn}$. The sum of the elements in the leading diagonal of A is called the **trace** of the matrix, denoted by tr(A).

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \qquad \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

- A square matrix in which all the elements below the leading diagonal are zero is called an **upper triangular matrix**, often denoted by U.

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & \dots & u_{1n} \\ 0 & u_{22} & \dots & \dots & u_{2n} \\ 0 & 0 & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & u_{nn} \end{bmatrix} \qquad u_{ij} = 0 \quad \text{when} \quad i > j$$

- A square matrix in which all the elements above the leading diagonal are zero is called a **lower triangular matrix**, often denoted by L.

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & 0 \\ l_{n1} & l_{n2} & \vdots & \dots & l_{nn} \end{bmatrix} \qquad l_{ij} = 0 \quad \text{when} \quad i < j$$

- A square matrix where all the non-zero elements are along the leading diagonal is called a diagonal matrix, often denoted by D.

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & d_{nn} \end{bmatrix} \qquad d_{ij} = 0 \quad \text{when} \quad i \neq j$$

Some examples of matrices and their classification

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 2×3. It is not square.

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is 2×2. It is square.

Also, tr(A) does not exist, and tr$(B) = 1 + 4 = 5$.

$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ are both 3×3, square and upper triangular.

Also, tr$(C) = 0$ and tr$(D) = 3$.

$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ are both 3×3, square and lower triangular.

Also, tr$(E) = 0$ and tr$(F) = 4$.

$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are both 3×3, square and diagonal.

Also, tr$(G) = 0$ and tr$(H) = 6$.

Task

Classify the following matrices (and, where possible, find the trace):

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -3 & -2 & -4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Your solution

Answer

A is 3×2, B is 3×4, C is 4×4 and square.

The trace is not defined for A or B. However, tr$(C) = 34$.

Task

Classify the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your solution

Answer

A is 3×3 and square, B is 3×3 lower triangular, C is 3×3 upper triangular and D is 3×3 diagonal.

Equality of matrices

As we noted earlier, the terms in a matrix are called the **elements** of the matrix.

The elements of the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$ are $1, 2, -1, -4$

We say two matrices A, B are **equal** to each other only if A and B have the same number of rows and the same number of columns and if each element of A is equal to the corresponding element of B. When this is the case we write $A = B$. For example if the following two matrices are equal:

$$A = \begin{bmatrix} 1 & \alpha \\ -1 & -\beta \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

then we can conclude that $\alpha = 2$ and $\beta = 4$.

The unit matrix

The **unit matrix** or the **identity matrix**, denoted by I_n (or, often, simply I), is the diagonal matrix of order n in which all diagonal elements are 1.

Hence, for example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The zero matrix

The **zero matrix** or **null matrix** is the matrix all of whose elements are zero. There is a zero matrix for every size. For example the 2×3 and 2×2 cases are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Zero matrices, of whatever size, are denoted by $\underline{0}$.

The transpose of a matrix

The **transpose** of a matrix A is a matrix where the rows of A become the columns of the new matrix and the columns of A become its rows. For example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The resulting matrix is called the **transposed matrix** of A and denoted A^T. In the previous example it is clear that A^T is not equal to A since the matrices are of different sizes. If A is square $n \times n$ then A^T will also be $n \times n$.

Example 1

Find the transpose of the matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution

Interchanging rows with columns we find

$$B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Both matrices are 3×3 but B and B^T are clearly different.

When the transpose of a matrix is equal to the original matrix i.e. $A^T = A$, then we say that the matrix A is **symmetric**. (This is because it has symmetry about the leading diagonal.)
In Example 1 B is **not** symmetric.

Example 2

Show that the matrix $C = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

Solution

Taking the transpose of C:

$$C^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}.$$

Clearly $C^T = C$ and so C is a symmetric matrix. Notice how the leading diagonal acts as a "mirror"; for example $c_{12} = -2$ and $c_{21} = -2$. In general $c_{ij} = c_{ji}$ for a symmetric matrix.

Task

Find the transpose of each of the following matrices. Which are symmetric?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Your solution

Answer

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = C, \text{ symmetric}$$

$$D^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} \quad E^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E, \text{ symmetric}$$

3. Addition and subtraction of matrices

Under what circumstances can we add two matrices i.e. define $A + B$ for given matrices A, B?

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 5 & 6 & 9 \\ 7 & 8 & 10 \end{bmatrix}$$

There is no sensible way to define $A + B$ in this case since A and B are different sizes.
However, if we consider matrices of the same size then addition can be defined in a very natural way. Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. The 'natural' way to add A and B is to add corresponding elements together:

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

In general if A and B are both $m \times n$ matrices, with elements a_{ij} and b_{ij} respectively, then their sum is a matrix C, also $m \times n$, such that the elements of C are

$$c_{ij} = a_{ij} + b_{ij} \qquad i = 1, 2, \ldots, m \quad j = 1, 2, \ldots, n$$

In the above example

$$c_{11} = a_{11} + b_{11} = 1 + 5 = 6 \qquad c_{21} = a_{21} + b_{21} = 3 + 7 = 10 \quad \text{and so on.}$$

Subtraction of matrices follows along similar lines:

$$D = A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

4. Multiplication of a matrix by a number

There is also a natural way of defining the product of a matrix with a number. Using the matrix A above, we note that

$$A + A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

What we see is that $2A$ (which is the shorthand notation for $A + A$) is obtained by multiplying *every* element of A by 2.
In general if A is an $m \times n$ matrix with typical element a_{ij} then the product of a number k with A is written kA and has the corresponding elements ka_{ij}.

Hence, again using the matrix A above,

$$7A = 7\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix}$$

Similarly:

$$-3A = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix}$$

Task

For the following matrices find, where possible, $A+B,\ A-B,\ B-A,\ 2A$.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Your solution

Answer

1. $A+B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad A-B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad B-A = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix} \qquad 2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

2. $A+B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 8 & 9 & 10 \end{bmatrix} \qquad A-B = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 6 & 7 \\ 6 & 7 & 8 \end{bmatrix} \qquad B-A = \begin{bmatrix} 0 & -1 & -2 \\ -5 & -6 & -7 \\ -6 & -7 & -8 \end{bmatrix}$

   $2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

3. None of $A+B,\ A-B,\ B-A$, are defined. $\qquad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$

5. Some simple matrix properties

Using the definition of matrix addition described above we can easily verify the following properties of matrix addition:

Key Point 2

Basic Properties of Matrices

Matrix addition is **commutative**: $A + B = B + A$

Matrix addition is **associative**: $A + (B + C) = (A + B) + C$

The **distributive law** holds: $k(A + B) = k\,A + k\,B$

These Key Point results follow from the fact that $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ etc.

We can also show that the transpose of a matrix satisfies the following simple properties:

Key Point 3

Properties of Transposed Matrices

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ (A-B)^T &= A^T - B^T \\ (A^T)^T &= A \end{aligned}$$

Example 3

Show that $(A^T)^T = A$ for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Solution

$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ so that $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$

Task

For the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $3(A+B) = 3A + 3B$ (ii) $(A-B)^T = A^T - B^T$.

Your solution

Answer

(i) $A + B = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$; $3(A+B) = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$; $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$;

$3B = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$; $3A + 3B = \begin{bmatrix} 6 & 3 \\ 6 & 15 \end{bmatrix}$.

(ii) $A - B = \begin{bmatrix} 0 & 3 \\ 4 & 3 \end{bmatrix}$; $(A-B)^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$;

$B^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $A^T - B^T = \begin{bmatrix} 0 & 4 \\ 3 & 3 \end{bmatrix}$.

Exercises

1. Find the coefficient matrix A of the system:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ 4x_1 + 4x_2 &= 0 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

If $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ determine $(3A^T - B)^T$.

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$ verify that $3(A^T - B) = (3A - 3B^T)^T$.

Answers

1. $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & 0 \\ 2 & -1 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 4 & -1 \\ -1 & 0 & -1 \end{bmatrix}, \quad 3A^T = \begin{bmatrix} 6 & 12 & 6 \\ 9 & 12 & -3 \\ -3 & 0 & -3 \end{bmatrix}$

$3A^T - B = \begin{bmatrix} 5 & 10 & 3 \\ 5 & 7 & -9 \\ -3 & 0 & -4 \end{bmatrix} \quad (3A^T - B)^T = \begin{bmatrix} 5 & 5 & -3 \\ 10 & 7 & 0 \\ 3 & -9 & -4 \end{bmatrix}$

2. $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad A^T - B = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}, \quad 3(A^T - B) = \begin{bmatrix} 6 & 0 \\ 6 & 12 \\ 3 & -3 \end{bmatrix}$

$B^T = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & 7 \end{bmatrix}, \quad 3A - 3B^T = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 6 \\ 12 & 3 & 21 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 0 & 12 & -3 \end{bmatrix}$